Dense Orbits of a Two Point Algebra

A. Andonie and D. Lara

Abstract - Finite dimensional algebras over a field can be classified into three classes by representation type. These are finite, tame, and wild. Wild algebras contain copies of the representations of all finite dimensional algebras so classifying the indecomposable representations is not feasible. We then look at representations whose orbit is dense in their associated irreducible component of their representation variety. With this geometric perspective, we can approximate our representations. We give an example of a two point dense orbit algebra of wild representation type and show that it has a dense orbit for dimension vectors of certain types.

Keywords : quiver; quiver representation; finite dimensional algebra; representation type; dense orbit

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1 Introduction

1.1 Context and Motivation

The representation types of algebras have been classified by the Trichotomy Theorem of Drozd, [2]. These are

1. Finite: Up to isomorphism, there are finitely many representations.
2. Tame: For each fixed dimension d, there exist finitely many families of representations that depend on at most one parameter.
3. Wild: Given an arbitrary positive integer N, there exists a family of representations that depend on N parameters.

From this description, classifying the representations of wild finite dimensional algebras is not possible. However, there is a way to "approximate" the representations for some wild algebras by looking at the indecomposable representations that yield a dense orbit. A dense orbit algebra is an algebra that has a dense orbit in each irreducible component of its representation variety for each dimension vector. It is conjectured that every dense orbit algebra has finitely many indecomposable representations whose orbits are dense in their respective irreducible components. If this conjecture is true, classifying the representations whose orbits are open in their respective subvarieties becomes a finite problem.

1.2 Main Result

The goal of this project is to broaden the example base of dense orbit Algebras and the techniques in how to find them. The main result is as follows.

Theorem 1.1 *Let Λ be the algebra given by the following quiver and relations.*

$$\begin{tikzcd} \underset{1}{\bullet} \arrow[loop left, "a"] \arrow[r, "c"] & \underset{2}{\bullet} \arrow[loop right, "b"] \end{tikzcd} \qquad a^7 = b^3 = ca - bc = 0 \tag{1}$$

This is an algebra of wild representation type and each irreducible component of the representation variety $\operatorname{rep}_\Lambda(\mathbf{d})$ *has a dense orbit for dimension vectors of type* $\mathbf{d} = (\mathbf{p}, \mathbf{q}) = (p_1 + p_2 + p_3, q_1 + q_2 + q_3)$ *where* $p_i, q_j \in \mathbb{Z}_{n>0}$ *with* $p_1 > p_2 > p_3$*,* $q_1 > q_2 > q_3$ *and* $\mathbf{p}$*,* $\mathbf{q}$ *have parts of sizes at most* 7 *and* 3 *respectively.*

This algebra has been shown to be of wild representation type by Hoshino and Miyachi [3]. We conjecture that the algebra Λ has a dense orbit for all dimension vector $\mathbf{d}$ but a complete proof does not seem within reach at the moment.

2 Background

For a more in-depth review of quiver algebras and their representations, we refer the reader to [1]. Quivers and quiver representations are tools that let us have a visual representation of an algebra and its modules. Every finite dimensional algebra over an algebraically closed field corresponds to a quiver with relations and the modules of the algebras correspond to the representations of the quiver. Therefore, we will define everything in terms of quivers and their representations.

2.1 Quivers and Quiver Representations

A quiver is a directed graph. More formally, a quiver is a quadruple $Q = (Q_0, Q_1, s, t)$ where Q_0 is the set of vertices, Q_1 is the set of arrows and $s, t : Q_1 \to Q_0$ are functions from the set of arrows to the set of vertices. That is, a quiver is a directed graph where loops and multiple arrows from a vertex are allowed. The maps s, t state where the arrows start and end. Given $\alpha \in Q_1$, $s(\alpha)$ is called the source of the arrow and $t(\alpha)$ is called the target of the arrow.
For the rest of this article, assume that $\mathbb{k}$ is an algebraically closed field. Given a quiver Q, we define a *quiver representation* by placing $\mathbb{k}$-vector spaces in each of the vertices and linear maps on the arrows. That is, if $Q = (Q_0, Q_1, s, t)$ then a quiver representation M of Q is a collection $(M_i, \varphi_\alpha)_{\{i \in Q_0, \alpha \in Q_1\}}$ where each M_i is a $\mathbb{k}$-vector space and $\varphi_\alpha : M_{s(\alpha)} \to M_{t(\alpha)}$ is a linear map. A *path* σ is defined to be a sequence of arrows

$$\sigma = \alpha_n \cdots \alpha_1,$$

where $t(\alpha_i) = s(\alpha_{i+1})$. We also include the trivial paths e_i at each of the vertices. For a non-trivial path $\sigma = \alpha_n \cdots \alpha_1$, an *evaluation* of a representation M on a path σ is defined to be

$$\varphi_\sigma = \varphi_{\alpha_n} \circ \cdots \circ \varphi_{\alpha_1}.$$

A *relation* is a linear combination of paths. A representation $M = (M_i, \varphi_\alpha)_{i \in Q_0, \alpha \in Q_1}$ is said to be bound by a relation $\sum_{i=1}^n a_i \sigma_i$ if

$$\sum_{i=1}^n a_i \varphi_{\sigma_i} = 0.$$

We now define the main algebraic object of interest, the *quiver path algebra*. A quiver path algebra $\Bbbk Q$ of a quiver Q is a vector space that is formed by the set of all linear combinations of paths and multiplication of paths is defined as concatenation of paths. That is, given two paths $\sigma = \alpha_n \cdots \alpha_1, \omega = \beta_m \cdots \beta_1$ the product is then defined as

$$\omega\sigma = \beta_m \cdots \beta_1 \alpha_n \cdots \alpha_1$$

when $t(\alpha_n) = s(\beta_1)$ and 0 otherwise. The ideal R_Q is the arrow ideal of the path algebra $\Bbbk Q$. A two sided ideal I is said to be admissible if there exists $n \geq 2$ such that

$$R_Q^n \subseteq I \subseteq R_Q^2.$$

Given a finite dimensional algebra A, we can construct a quiver whose path algebra (mod an admissible ideal) is isomorphic to A.

Theorem 2.1 *For every $\Bbbk$-algebra A there is an associated quiver Q and an admissible ideal I of $\Bbbk Q$ such that*

$$A \cong \Bbbk Q/I$$

In addition, it is known that every finite dimensional algebra over an algebraically closed field corresponds to a bound quiver algebra $\Bbbk Q/I$ whose representations categories are equivalent. For the rest of the article, let $A \cong \Bbbk Q/I$ be a bound quiver algebra.

We next define the direct sum of representations for a bound quiver Q with relations R. Given two quiver representations $M = (M_i, \varphi_\alpha)$ $N = (N_i, \varphi_\alpha)$, the direct sum of M and N is defined to be

$$M \oplus N := ((M_i \oplus N_i, \varphi_\alpha \oplus \varphi_\alpha).$$

A representation is *indecomposable* if it cannot be written as a non-trivial direct sum. Given a dimension vector $\mathbf{d}\colon Q_0 \to \mathbb{Z}_{\geq 0}$, we study the *representation variety*

$$\operatorname{rep}_Q(\mathbf{d}) = \prod_{\alpha \in Q_1} \operatorname{Mat}_{\mathbf{d}(h\alpha), \mathbf{d}(t\alpha)}(\Bbbk),$$

where $\operatorname{Mat}_{m,n}(\Bbbk)$ denotes the space of matrices with m rows, n columns, and entries in the field $\Bbbk$. This representation variety is a collection of representations of the quiver Q. The base change group

$$\mathbf{GL}(\mathbf{d}) = \prod_{x \in Q_1} \mathbf{GL}(d(x))$$

acts on the variety by simultaneous conjugation. That is,

$$g \cdot M := (g_{h\alpha} M_\alpha g_{t\alpha}^{-1})_{\alpha \in Q_1},$$

This is the collection of all representation that are isomorphic as quiver representations to M. An orbit is said to be dense in its irreducible component if its closure (under the Zariski topology) is equal to the irreducible component. We say that an algebra is a *dense orbit algebra* if each irreducible component of the variety $\operatorname{rep}_A(\mathbf{d})$ has a dense orbit for each and every dimension vector $\mathbf{d}$.

2.2 An Example

We illustrate our definitions with an example. Consider the quiver Q

$$\begin{tikzcd} \underset{1}{\bullet} \arrow[loop left, "\alpha"] \end{tikzcd}$$

with relation

$$\alpha^2 = 0.$$

Our set of vertices is $Q_0 = \{1\}$ and the set of arrows is $Q_1 = \{\alpha\}$. We then define the maps that tell us where the arrow starts and ends. t, s:$Q_1 \to Q_0$ are defined as $s(\alpha) = 1$, $t(\alpha) = 1$.
The quiver path algebra is a familiar object. The basis is taken to be the set of all paths that are bound by the relation. These are:

$$e_1, \alpha$$

The path algebra is isomorphic to $\Bbbk[T]/(T^2)$. Let A denote the path algebra. Let M be the following quiver representation.

$$\begin{tikzcd} \Bbbk^2 \arrow[loop left, "\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}"] \end{tikzcd}$$

The representation variety of dimension vector 2 is equal to set of all 2 by 2 matrices whose square is equal to 0.

$$\operatorname{rep}_A(2) = \{A \in Mat_{2,2}(\Bbbk) \mid A^2 = 0\}$$

The group $\mathbf{GL}(2)$ acts on $\operatorname{rep}_A(2)$ by conjugation

$$g \cdot A := gAg^{-1}.$$

The orbit of the representation M is the set of isomorphic representations.

$$O(M) = \left\{ g \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} g^{-1} \mid g \in \mathbf{GL}(2) \right\}$$

In familiar terms, this is the set of all matrices A that are similar to $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. This algebra is of finite representation type and it is known that all representation finite algebras are dense orbit algebras. It is often difficult to show that the orbit is dense by brute force. While we still rely on computations, we redefine our problem in terms of polynomials to make the computations more feasible.

3 Proof of the Main Result

3.1 Reductions

The method of proof involves finding an open condition the orbit by direct computation. We reframe the problems in terms of polynomials. Fix a $d_1 \times d_1$ matrix A and a $d_2 \times d_2$ matrix B with entries over $\Bbbk$. We let $\mathbf{p}$ and $\mathbf{q}$ denote a partion of d_1 and d_2 with parts at most 7 and 3 respectively. We can then identify each pair $(A, \Bbbk^{d_1})$ $(B, \Bbbk^{d_2})$ with $\Bbbk[T]$-modules X and Y respectively. A and B are similar to their corresponding Jordan forms of eigenvalue zero. The partitions $\mathbf{p} = (p_1, p_2, \cdots)$ and $\mathbf{q} = (q_1, q_2, \cdots)$ then correspond to the size of the Jordan blocks from largest to smallest. Furthermore, each Jordan block can be identified with the module $J_k := \Bbbk[T]/(T^k)$ and hence,

$$X \cong \bigoplus_{j=1}^{l} J_{p_j}$$

$$X \cong \bigoplus_{i=1}^{m} J_{q_i}$$

The relation $ca - bc = 0$ lets us identify C as a $\Bbbk[T]$-module morphism. The ith row is labeled by J_{q_i} and jth column is labeled J_{p_j}. The entry of the matrix C in the row labeled J_l and column J_k represents is a $\Bbbk[T]$-module morphism $J_k \to J_l$ so it is represented by a polynomial f. Furthermore, the relation $b^3c = 0$ requires that f is annihilated by T^3. Thus F can be taken of the form below, where $a, b, c \in \Bbbk$

$$f = \begin{cases} a & \text{if } l = 1 \\ a + bT & \text{if } l = 2 \\ aT^{l-1} & \text{if } k = 1 \\ aT^{l-1} + bT^{l-2} & \text{if } k = 2 \\ aT^{l-1} + bT^{l-2} + cT^{l-3} & \text{if } k, l \geq 3 \end{cases}$$

The action on the variety corresponds to an action of $Aut_{\Bbbk[T]}(X) \times Aut_{\Bbbk[T]}(Y)$ on $\mathrm{Hom}_{\Bbbk[T]}(X, Y)$ which translates to row and column operations on our $\Bbbk[T]$-labeled matrix.

Lemma 3.1 *The following operations correspond to actions of elements of the group* $\mathrm{Aut}_{\Bbbk[T]}(X) \times \mathrm{Aut}_{\Bbbk[T]}(Y)$ *on* $\mathrm{Hom}_{\Bbbk[T]}(X, Y)$*:*

(i) multiplication of a row labeled J_i *by an invertible element of* $\Bbbk[T]/(T^i)$*, and similarly for columns;*

(ii) row operations replacing a row labeled J_j *with the sum of* J_j *and* f *times a row labeled* J_i*, where* $f \in \Bbbk[T]$ *if* $j \leq i$ *and* $f \in (T^{j-i})$ *if* $j > i$*;*

(iii) column operations replacing a column labeled J_j *with the sum of* J_j *and* f *times a column labeled* J_i*, where* $f \in \Bbbk[T]$ *if* $j \geq i$ *and* $f \in (T^{i-j})$ *if* $j < i$*.*

Our main strategy is to perform row and column operations as described in Lemma 3.1 with the additional results. Lemma 3.6 from [4] establishes the general forms of our irreducible components and therefore we can pick an arbitrary $\Bbbk[T]$ labeled matrix of partition $(\mathbf{q}, \mathbf{p})$ and get an open condition via our row and column operations. We use Corollary 2.2 from [4] that justifies a reduction where it is sufficient to show we either have a dense orbit or show that a direct sum of the representation has a dense orbit.

3.2 Proof of Main Result

We will prove our result by proving the following series of lemmas.

Lemma 3.2 $\mathrm{rep}_\Lambda(\mathbf{d})$ *has a dense orbit for all dimension vectors* $\mathbf{d} = (d_1, d_2)$ *if there exist* i, j *such that* $p_i = q_j = 3$ $\mathrm{rep}_\Lambda(\mathbf{d})$.

Proof. Suppose that for some i, j, $p_i = q_j = 3$. A matrix $C \in \mathrm{Hom}_{[T]}(X, Y)$ then has the form below.

$$\begin{array}{c} \\ \vdots \\ J_3 \\ \vdots \end{array} \begin{array}{c} \begin{array}{ccc} \cdots & J_3 & \cdots \end{array} \\ \begin{pmatrix} * & * & * \\ * & a + bT + CT^2 & * \\ * & * & * \end{pmatrix} \end{array}$$

In the i, j entry, we first perform a row operation $gR_i \to R_i$ where $g(a + bT + cT^2) = 1$.

$$\begin{array}{c} \\ \vdots \\ J_3 \\ \vdots \end{array} \begin{array}{c} \begin{array}{ccc} \cdots & J_3 & \cdots \end{array} \\ \begin{pmatrix} * & * & * \\ * & a + bT + CT^2 & * \\ * & * & * \end{pmatrix} \end{array} \to \begin{array}{c} \\ \vdots \\ J_3 \\ \vdots \end{array} \begin{array}{c} \begin{array}{ccc} \cdots & J_3 & \cdots \end{array} \\ \begin{pmatrix} * & * & * \\ * & 1 & * \\ * & * & * \end{pmatrix} \end{array}$$

Next, we zero out the terms above this entry with a row operation of type $fR_{J_i} + R_{J_k} \to R_k$ where $f \in (T^{p_k - p_i})$ where $p_i \leq p_k$. This is possible because the terms in row R_k must be of the form $aT^{p_k-3} + bT^{p_k-2} + cT^{p_k-1}$. In addition, we zero out the terms to the left, below, and to the right of this entry with appropriate row and column operations.

$$\begin{array}{cc} & \begin{array}{ccc} \cdots & J_3 & \cdots \end{array} \\ \begin{array}{c} \vdots \\ J_3 \\ \vdots \end{array} & \begin{pmatrix} * & * & * \\ * & 1 & * \\ * & * & * \end{pmatrix} \end{array} \to \begin{array}{cc} & \begin{array}{ccc} \cdots & J_3 & \cdots \end{array} \\ \begin{array}{c} \vdots \\ J_3 \\ \vdots \end{array} & \begin{pmatrix} * & 0 & * \\ 0 & 1 & 0 \\ * & 0 & * \end{pmatrix} \end{array}$$

In this case, $\begin{array}{cc} & J_3 \\ J_3 & \begin{pmatrix} 1 \end{pmatrix} \end{array}$ splits off. □

Lemma 3.3 *For a dimension vector* $\mathbf{d} = (d_1, d_2)$*, if there exist* i, j *such that* $p_i = q_j = 2$ *for some* i, j*, then* $\mathrm{rep}_\Lambda(\mathbf{d})$ *has a dense orbit for each irreducible component.*

Proof. If there exist i, j such that $p_i = q_j = 2$, then our matrix C has the following form with a polynomial of type $a + bT$ in the i, j entry.

$$\begin{array}{cc} & \begin{array}{ccc} \cdots & J_2 & \cdots \end{array} \\ \begin{array}{c} \vdots \\ J_2 \\ \vdots \end{array} & \begin{pmatrix} * & * & * \\ * & a + bT & * \\ * & * & * \end{pmatrix} \end{array}$$

Using a similar process to the previous lemma, we first get a 1 entry with a row automorphism. We then move to zero out the terms above this entry which are of the form $aT^{p_k-2} + bT^{p_k-1}$ with $p_k \geq p_i$. Thus we use row operations of type $fR_i + R_k \to R_k$ where $f \in (T^{p_k - p_i})$. We can clear out all terms to the left of the (i, j) since we do not have any restrictions doing column operations from right to left. In the other direction, we must do column operations of type $gC_{q_j} + C_{q_r} \to C_{q_r}$ where $g \in (T^{q_j - q_r})$, but since $q_j = 2$ and $q_r = 1$ or $q_r = 2$, it is possible to zero out all the terms to the right. Lastly, we zero out the terms below this entry. Row operations in this direction have no restrictions.

$$\begin{array}{cc} & \begin{array}{ccc} \cdots & J_2 & \cdots \end{array} \\ \begin{array}{c} \vdots \\ J_2 \\ \vdots \end{array} & \begin{pmatrix} * & * & * \\ * & 1 & * \\ * & * & * \end{pmatrix} \end{array} \to \begin{array}{cc} & \begin{array}{ccc} \cdots & J_2 & \cdots \end{array} \\ \begin{array}{c} \vdots \\ J_2 \\ \vdots \end{array} & \begin{pmatrix} * & 0 & * \\ 0 & 1 & 0 \\ * & 0 & * \end{pmatrix} \end{array}$$

Thus the representation with a dense orbit, $\begin{array}{cc} & J_3 \\ J_3 & \begin{pmatrix} 1 \end{pmatrix} \end{array}$, splits off. □

Lemma 3.4 *For a dimension vector* $\mathbf{d}$ *where* $p_i = q_j = 1$ *for some* i, j*,*

Proof. When $p_i = q_j = 1$

$$\begin{array}{cc} & \begin{array}{ccc} \cdots & J_1 & \cdots \end{array} \\ \begin{array}{c} \vdots \\ J_1 \\ \vdots \end{array} & \begin{pmatrix} * & * & * \\ * & a & * \\ * & * & * \end{pmatrix} \end{array} \to \begin{array}{cc} & \begin{array}{ccc} \cdots & J_1 & \cdots \end{array} \\ \begin{array}{c} \vdots \\ J_1 \\ \vdots \end{array} & \begin{pmatrix} * & * & * \\ * & 1 & * \\ * & * & * \end{pmatrix} \end{array}$$

$$\begin{array}{cc} & \begin{array}{ccc} \cdots & J_1 & \cdots \end{array} \\ \begin{array}{r} \vdots \\ \to J_1 \\ \vdots \end{array} & \begin{pmatrix} * & 0 & * \\ 0 & 1 & 0 \\ * & 0 & * \end{pmatrix} \end{array}$$

Repeating the process of clearing the terms around this entry, the possible terms to the left and above are aT^{j-1} or a. The terms to the right and below have the form a. In this case, there is a 1 by 1 block that splits off. □

In each of these 3 lemmas, we observe that we have a 1 by 1 block that splits off. Note that we did not assume anything about the lengths of the partitions so these results hold in general. We will use these results to prove the following. For each lemma below, assume the partitions $\mathbf{p}$ and $\mathbf{q}$ satisfy the assumptions as in Theorem 1.1.

Lemma 3.5 *If $l(\mathbf{q}) = 1$ and $\mathbf{d} = (d_1, d_2) = (\sum_{i=1}^{3} p_i, \sum_{j=1}^{3} q_j)$, then $\mathrm{rep}_\Lambda(\mathbf{d})$ has a dense orbit in each irreducible component.*

Proof. The representation matrix C then has the following form.

$$\begin{array}{cc} & J_{q_1} \\ \begin{array}{c} J_{p_1} \\ J_{p_2} \\ J_{p_3} \end{array} & \begin{pmatrix} * \\ * \\ * \end{pmatrix} \end{array}$$

For notational simplicity, we use T^* to denote a term in a row or column that has no parameters. From 3 distinct row automorphisms, we can reduce the matrix into the form

$$\begin{array}{cc} & J_{q_1} \\ \begin{array}{c} J_{p_1} \\ J_{p_2} \\ J_{p_3} \end{array} & \begin{pmatrix} T^* \\ T^* \\ T^* \end{pmatrix} \end{array}$$

and see that we have a dense orbit in this case. □

Lemma 3.6 *If $l(\mathbf{q}) = 2$ and $\mathbf{d} = (d_1, d_2) = (\sum_{i=1}^{3} p_i, \sum_{j=1}^{3} q_j)$, then $\mathrm{rep}_\Lambda(\mathbf{d})$ has a dense orbit in each irreducible component.*

Proof. The representation matrix C then has the following form.

$$\begin{array}{cc} & \begin{array}{cc} J_{q_2} & J_{q_1} \end{array} \\ \begin{array}{c} J_{p_1} \\ J_{p_2} \\ J_{p_3} \end{array} & \begin{pmatrix} * & * \\ * & * \\ * & * \end{pmatrix} \end{array}$$

We will start with the lower right most entry. We will have 3 cases that depend on the value of q_1.

If $q_1 = 1$, then we can assume that $p_i \geq 2$ for all i by Lemma 3.2. Then using a row automorphism on the last row, we have reduced our matrix as follows.

$$\begin{array}{cc} & \begin{array}{cc} J_{q_2} & J_1 \end{array} \\ \begin{array}{c} J_{p_1} \\ J_{p_2} \\ J_{p_3} \end{array} & \begin{pmatrix} * & * \\ * & * \\ * & T^{p_3-1} \end{pmatrix} \end{array}$$

We will zero out the terms above this entry with row operations $gR_{p_3} + R_{p_i} \to R_{p_i}$ where $g \in (T^{p_i-p_3})$. The entries above this entry are of the form aT^{p_i-1} for some $a \in \Bbbk$ so we can let $g = -aT^{p_i-p_3}$.

$$\begin{array}{cc} & \begin{array}{cc} J_{q_2} & J_1 \end{array} \\ \begin{array}{c} J_{p_1} \\ J_{p_2} \\ J_{p_3} \end{array} & \begin{pmatrix} * & 0 \\ * & 0 \\ * & T^{p_3-1} \end{pmatrix} \end{array}$$

Next, we do a column automorphism to get rid of the parameters in the $(3,2)$ entry.

$$\begin{array}{cc} & \begin{array}{cc} J_{q_2} & J_1 \end{array} \\ \begin{array}{c} J_{p_1} \\ J_{p_2} \\ J_{p_3} \end{array} & \begin{pmatrix} * & 0 \\ * & 0 \\ T^* & T^{p_3-1} \end{pmatrix} \end{array}$$

and thus we can do 2 more row automorphisms to get an open condition in this case.
In the next case when $q_1 = 2$, we can assume that $p_i \neq 2$ for all i. Since our partitions do not repeat, we have $\mathbf{p} = (p_1, p_2)$. If $p_2 = 1$, then we start with the $(2,2)$ entry with a row automorphism.

$$\begin{array}{cc} & \begin{array}{cc} J_{q_2} & J_2 \end{array} \\ \begin{array}{c} J_{p_1} \\ J_{p_2} \end{array} & \begin{pmatrix} * & * \\ * & * \end{pmatrix} \end{array} \to \begin{array}{cc} & \begin{array}{cc} J_{q_2} & J_2 \end{array} \\ \begin{array}{c} J_{p_1} \\ J_{p_2} \end{array} & \begin{pmatrix} * & * \\ * & 1 \end{pmatrix} \end{array}$$

We zero out the $(2,1)$ entry with a column operation

$$\begin{array}{cc} & \begin{array}{cc} J_{q_2} & J_2 \end{array} \\ \begin{array}{c} J_{p_1} \\ J_{p_2} \end{array} & \begin{pmatrix} * & * \\ * & * \end{pmatrix} \end{array} \to \begin{array}{cc} & \begin{array}{cc} J_{q_2} & J_2 \end{array} \\ \begin{array}{c} J_{p_1} \\ J_{p_2} \end{array} & \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \end{array}$$

and lastly, we do a row automorphism followed by a column operation to get rid of all remaining parameters to get an open condition on the representation.

$$\begin{array}{cc} & \begin{array}{cc} J_{q_2} & J_2 \end{array} \\ \begin{array}{c} J_{p_1} \\ J_{p_2} \end{array} & \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \end{array} \to \begin{array}{cc} & \begin{array}{cc} J_{q_2} & J_2 \end{array} \\ \begin{array}{c} J_{p_1} \\ J_{p_2} \end{array} & \begin{pmatrix} T^* & 1 \\ 0 & 1 \end{pmatrix} \end{array}$$

□

Lemma 3.7 *If $l(\mathbf{q}) = 3$ and $\mathbf{d} = (d_1, d_2) = (\sum_{i=1}^{3} p_i, \sum_{j=1}^{3} q_j)$, then $\text{rep}_\Lambda(\mathbf{d})$ has a dense orbit in each irreducible component.*

Proof. Since our partitions do not repeat, the representation matrix C has the form

$$\begin{array}{c} \\ J_{p_1} \\ J_{p_2} \\ J_{p_3} \end{array} \begin{array}{c} \begin{array}{ccc} J_3 & J_2 & J_1 \end{array} \\ \begin{pmatrix} * & * & * \\ * & * & * \\ * & * & * \end{pmatrix} \end{array}$$

By Lemmas 3.2, 3.3, and 3.4, it must be that $p_i \geq 4$ for all i. We start with a row automorphsim in the $(3,1)$ entry to get rid of the parameters.

$$\begin{array}{c} \\ J_{p_1} \\ J_{p_2} \\ J_{p_3} \end{array} \begin{array}{c} \begin{array}{ccc} J_3 & J_2 & J_1 \end{array} \\ \begin{pmatrix} * & * & * \\ * & * & * \\ * & * & * \end{pmatrix} \end{array} \rightarrow \begin{array}{c} \\ J_{p_1} \\ J_{p_2} \\ J_{p_3} \end{array} \begin{array}{c} \begin{array}{ccc} J_3 & J_2 & J_1 \end{array} \\ \begin{pmatrix} * & * & * \\ * & * & * \\ T^{p_3-1} & * & * \end{pmatrix} \end{array}$$

We then zero out the terms to right of this entry. This is possible since the terms to the right of this entry are either of the type $aT^{p_3-2} + bT^{p_3-1}$ or cT^{p_3-1} for some $a, b, c \in \Bbbk$ and we are allowed to column operations of the type $gC_{q_3} + C_{q_i} \rightarrow C_{q_i}$ where $g \in (T^{q_3-q_i})$. Similarly, we also zero out the terms above the $(3,1)$ entry.

$$\begin{array}{c} \\ J_{p_1} \\ J_{p_2} \\ J_{p_3} \end{array} \begin{array}{c} \begin{array}{ccc} J_3 & J_2 & J_1 \end{array} \\ \begin{pmatrix} * & * & * \\ * & * & * \\ T^{p_3-1} & * & * \end{pmatrix} \end{array} \rightarrow \begin{array}{c} \\ J_{p_1} \\ J_{p_2} \\ J_{p_3} \end{array} \begin{array}{c} \begin{array}{ccc} J_3 & J_2 & J_1 \end{array} \\ \begin{pmatrix} 0 & * & * \\ 0 & * & * \\ T^{p_3-1} & 0 & 0 \end{pmatrix} \end{array}$$

and in this case we have that a representation that splits off. □

By the lemmas above, Theorem 1.1 is proven.

4 Future Work

Future work would automate the current reduction process by means of a computer program, thus speeding up the process by not doing the reductions by hand. At the moment, a program has already been built which can reduce small matrices with low dimension parameters, however it has yet to be fully completed. The program, written in Python, follows the reduction process shown in the previous section and reduces one entry at a time till the matrix cannot be reduced further. The program allows a user to enter the size of the matrix and the desired parameters. Once these are given, the program will proceed to reduce the matrix and show the user a step-by-step reduction of the matrix. This program can continue to be developed to reduce any size matrix with larger dimension parameters, notify the user if their input was invalid, and have a nicer user interface. If one is interested, one can find the Python code and file in [5]

We showed that we have a dense orbit for dimension vectors of certain size. We conjecture that this is a dense orbit algebra but a proof does not seem within reach at the moment. Though we have a strategy for certain partitions, it becomes very difficult to cover arbitrary partitions of type $p = (p_1, p_2, \cdots)$ and $q = (q_1, q_2, \cdots)$. Further work must be down to narrow down the partitions that give rise to irreducible components. We end with the following conjecture which we strongly suspect to be true based on a very similar problem done in [4].

Conjecture 4.1 Let $\Lambda(m, n)$ be the algebra given by the following quiver and relations.

$$\begin{tikzcd} \underset{1}{\bullet} \arrow[loop left, "a"] \arrow[rr, "c"] & & \underset{2}{\bullet} \arrow[loop right, "b"] \end{tikzcd} \qquad a^m = b^n = ca - bc = b^3c = 0 \tag{2}$$

This is an algebra of wild representation type except for certain small values of m and n and is a dense orbit algebra for all m, n.

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Adara Andonie
Central Washington University
Ellensburg, WA
E-mail: `adara.andonie@cwu.edu`

Danny Lara
Central Washington University
Ellensburg, WA
E-mail: `danny.lara@cwu.edu`

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