Self-Similar Structure of k- and Biperiodic Fibonacci Words

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Abstract - Defining the biperiodic Fibonacci words as a class of words over the alphabet $\{0, 1\}$, and two specializations the $k-$Fibonacci and classical Fibonacci words, we provide a self-similar decomposition of these words into overlapping words of the same type. These self-similar decompositions complement the previous literature where self-similarity was indicated but the specific structure of how the pieces interact was left undiscussed.

Keywords : fractals; Fibonacci words; biperiodic Fibonacci words; L-systems; iterated function systems

Mathematics Subject Classification (2020) **:** 68R15; 11B39

1 Introduction

The Fibonacci sequence is a classical numerical sequence defined by a recurrence relation in the additive semigroup of $\mathbb{N}$ by

$$F_1 = 0 \qquad F_2 = 1 \qquad F_n = F_{n-1} + F_{n-2}.$$

Consider instead the semigroup of words over an alphabet, $A = \{0, 1\}$. Let $A^* = \bigcup_{n \geq 0} A^n$ where the operation is concatenation. Then the sequence determined by the Fibonacci recurrence relation

$$f_1 = 0 \qquad f_2 = 01 \qquad f_n = f_{n-1} f_{n-2}$$

is called the Fibonacci sequence of words. For example:

$$f_3 = 010 \qquad f_4 = 01001 \qquad f_5 = 01001010.$$

Fibonacci words have appeared as examples in [8, 6] and their history is discussed in [1]. Two interesting features of Fibonacci words are that 1) there is an infinite word f such that every f_n appears as a prefix of f and 2) there is a substitution $\sigma(0) = 01$ and $\sigma(1) = 0$ such that $\sigma(f_n) = f_{n+1}$ and $\sigma(f) = f$. The existence of a substitution like σ is

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what is usually referred to as a word being *self-similar* [5]. This notion of self-similarity is a local notion, i.e. replace small pieces of an infinite word and the global word remains the same.

Fibonacci words and their generalizations (k-Fibonacci and biperiodic Fibonacci) that are introduced below are examples of Sturmian words [3, 9, 10]. Sturmian words have a structure theorem stating that a sequence of words is Sturmian if for some base cases and a sequence $\{q_n\}$ of positive integers then we have $s_{n+1} = s_n^{q_n - 1} s_{n-1}$ (see [3].) The Fibonacci words are generated when $q_n = 2$, the k-nacci words when $q_n = k + 1$, and the biperiodic Fibonacci words when q_n is a 2-periodic sequence. It is known [2] that a Sturmian word is self-similar if and only if the sequence $\{q_n\}_{n \geq 2}$ is periodic. So the words considered in this paper are the three simplest families of self-similar Sturmian words. The techniques used in this paper have been explored by the authors for 3-periodic sequence but that is not included as there was no new insight gained.

In Section 2 we define biperiodic Fibonacci words and cite many useful facts about them. We also define an *overlapping word* and prove several decomposition results using the overlapping word. Then in Section 3 we prove the main decomposition theorem about biperiodic Fibonacci words. Lastly, in Section 4 we address the edge cases where a and/or b equals one.

1.1 Self-similar Cell Structures

In [7] a *drawing rule* was introduced to automatically create a curve in $\mathbb{R}^2$ from f_n. The object of study there was the geometry of the drawn curves. These Fibonacci curves have scaling limits that are self-similar fractals. This paper is the first of a sequence investigating the self-similar fractal geometry of curves associated to generalizations of the Fibonacci words. This is done through viewing f_n as a concatenation (possibly with overlapping copies) of f_{n-l}. This is a large-scale notion of self-similarity where a macro object is viewed as comprising a small number of pieces, each of which is similar to the whole. In this paper we consider just the combinatorial properties of the bi-periodic Fibonacci words; the drawing rule, curves, and fractal geometry will be considered in subsequent papers.

We formalize the type of large-scale self-similarity we are seeking by the device of a *cell structure*. For example, consider the decomposition

$$f_n = f_{n-2} f_{n-2} f_{n-4} t_{n-2} t_{n-2}.$$

(The symbol t_n denotes f_n with the last two digits transposed, see Definition 2.5.) This can be read as "a Fibonacci word is composed of five subcells: four of size $n-2$ and one of size $n-4$." The important thing is that each of those subcells could again be subdivided using the same pattern providing *sub-subcells*. It will be seen below that it will be impossible to have subcells not overlap for $k-$Fibonacci words, let alone for biperiodic words. In the geometry of fractals having cells overlap is a normal occurrence and so this will be acceptable for us.

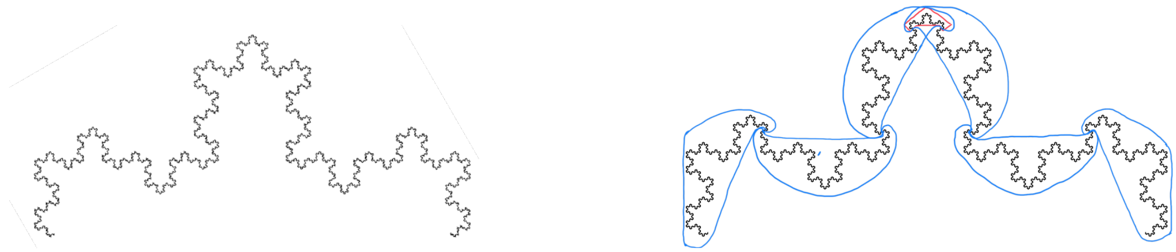

Figure 1: This is a 2-nacci curve using the drawing rule from [7]. The angle is $\pi/3$. The left hand picture is $n = 12$ and the right hand picture is $n = 14$. The pictures are very similar visually. Since $f_{2,14}$ begins with $f_{2,12}$ it is reasonable to ask where in the right hand picture the left hand one is. The lower right hand end of the right hand picture is the section corresponding to the leading copy of $f_{2,12}$ in $f_{2,14}$. However, one can see the same shape reoccur several times in these pictures. That is the geometric manifestation of the *cell structure* that is the topic of this paper.

Definition 1.1 *Let $\{f_n\}$ be a sequence of finite words on an alphabet Σ. We say that $\{f_n\}$ possesses a* self-similar cell structure *if there exists positive integers l and m such that for n of the form $n = n_0 + kl$ where $n_0 \geq ml$ there exists a decomposition of f_n that*

1. *uses only the subwords f_{n-l}, f_{n-2l}, $\ldots, f_{n-ml}$ and t_{n-l}, t_{n-2l}, $\ldots, t_{n-ml}$ (see Definition 2.5) whose indices are in the arithmetic progression,*

2. *the operations allowed for combining the subwords back into f_n are concatenation and overlapping concatenation where the overlaps can be described in terms of elements of $\{f_n\}$ but may be words with indices not in the arithmetic progression,*

3. *has the same pattern for all n in the arithmetic progression of n. (e.g. for all even $n \geq ml$); by the same pattern we mean the same number of each allowed subword i.e. t_{n-3l} and the same arrangement of the subwords independent of n as given by a formula in the paragraph below.*

We will call l the period of the cell structure. If $l = 1$ we call the cell structure trivial.

For example, we see from [7] that for the classic Fibonacci word,

$$f_n = f_{n-2}f_{n-2}f_{n-4}t_{n-2}t_{n-2}.$$

As will be seen in Theorem 3.4 we have that

$$f_{(2,2,n)} = f_{(2,2,n-2)}f_{(2,2,n-2)}I_{2,2,n-2)}t_{(2,2,n-2)}f_{(2,2,n-2)}$$

where $I_{(2,2,n-2)}$ is a copy of $f_{(2,2,n-2)}$ and of $t_{(2,2,n-2)}$ which overlap in a prescribed way; see Definition 2.8. These are examples of self-similar cell structures of period 2. Figure 1 shows geometric curves drawn from $f_{(2,2,n)}$ for various values of n. The ovals drawn over the graph correspond exactly to the subwords $f_{(2,2,n-2)}$ and $t_{(2,2,n-2)}$ identified in the cell structure. This correspondence to geometrical self-similar structures in the drawn curve and the combinatorial word decomposition is the reason for content of this definition.

It is not expected that a cell structure on a sequence of words will be unique. For example, if f_n can be decomposed into some overlapping concatenation of f_{n-2} and t_{n-2}'s then each of them could be decomposed again using the same cell structure so that now f_n is written as an overlapping concatenation of f_{n-4} and t_{n-4}'s. This is technically a different cell structure. It may also be possible for two different cell structures to be recognizable for different arithmetic progressions of n. For example in Theorem 3.4 we will see that we have different cell structures depending on whether n is even or odd in the $a \neq b$ cases.

Theorem 20 of [10] gives the periodicity of the substitution morphism σ to be $l = 2,\ 4,$ or 6, depending on the parities of the parameters a and b, for biperiodic Fibonacci words. This periodicity encodes the geometric self-similarity seen in Figure 1; The novel result of this paper is to capture this self-similarity in cell structures with periods matching that of σ. Theorem 3.4 provides such cell structures.

2 The Biperiodic Fibonacci Words

We begin with the definition of the sequence of biperiodic Fibonacci words and comment on how these become the k-nacci and classical Fibonacci words depending on the choice of a, b, and initial conditions. Then we will define a few related words. A few basic properties of these words will be quoted from [9] and [10]. As the words are defined using a biperiodic sequence, it will be convenient to have the following indexing shorthand.

Definition 2.1 *Let $a, b \geq 1$ and $n \geq 0$ be integers. Then we define*

$$r(a,b,n) = \begin{cases} a & \text{if } n \text{ is even} \\ b & \text{if } n \text{ is odd} \end{cases} \qquad s(a,b,n) = \begin{cases} b & \text{if } n \text{ is even} \\ a & \text{if } n \text{ is odd} \end{cases}$$

Although r, s technically depend on arguments a, b, and n, these parameters will generally be understood from context. Hence we will suppress the notation and simply write r, s.

Definition 2.2 *Let $a, b \geq 1$. The (a,b)-biperiodic Fibonacci words are defined by the recurrence relation*

$$f_{(a,b,0)} = 0 \qquad f_{(a,b,1)} = 0^{a-1}1 \qquad f_{(a,b,n)} = f^r_{(a,b,n-1)} f_{a,b,n-2}$$

For example, when $a = 2$ and $b = 3$, we have:

$$f_{(2,3,1)} = 01 \qquad f_{(2,3,2)} = 01\ 01\ 01\ 0 \qquad f_{(2,3,3)} = 0101010\ 0101010\ 01.$$

The cases where a or b are equal to 1 turn out to require more care and larger required n-values in most of the following results. *As a default for the rest of the paper we assume that $a, b \geq 2$.* The special considerations for when $a = 1$ or $b = 1$ are collected in Section 4.

Where to begin the indexing is a matter of choice. We choose primarily to begin our indexing at $n = 0$ to maintain agreement with the fractals produced in [9], while this is a shift in the index compared to [10]. Both articles are written by the same authors.

Definition 2.3 *The k-Fibonacci, or k-nacci, words are* $f_{k,n} = f_{(k,k,n)}$ *for* $k \geq 1$ *with the same initials conditions as in Definition 2.2.*

The classical Fibonacci words are $f_{n+1} = f_{(1,1,n)}$ *with the initial conditions* $f_{(1,1,0)} = 1$ *and* $f_{(1,1,1)} = 0$ *instead. Changing the initial conditions just exchanges the roles of* 0 *and* 1*, but not the cell structures, which only depend on the recurrence relation.*

We collect from [10] with a change of index some useful features and notation for biperiodic Fibonacci words.

Proposition 2.4 *Let* $f_{(a,b,n)}$ *be a biperiodic Fibonacci word. Then*

1. *For all* $n \geq 1$*, write* $f_{(a,b,n)} = p_{(a,b,n)}xy$*, so that* xy *are the last two characters. If* n *is odd then* $xy = 01$*, if* n *is even then* $xy = 10$*.*

2. *For all* $n \geq 1$ $p_{(a,b,n)}$ *is a palindrome.*

3. *For all* $n \geq 3$*,* $f_{(a,b,n-1)}p_{(a,b,n-2)} = f_{(a,b,n-2)}p_{(a,b,n-1)}$*.*

Recall that since we changed the indexing compared to [10], the indices here are correspondingly shifted.

Note that Proposition2.4.1 is not well-defined when $n = 0$, since we need $f_{(a,b,n)}$ to have at least two characters. For part 3, we need to consider $p_{(a,b,n-2)}$, so $n-2 \geq 1$; hence the $n \geq 3$ condition. Exceptions for small n are not worrisome as the drawn curves are considered as n grows towards infinity.

In light of Proposition 2.4 parts 1 and 2 it is reasonable to define a “co-biperiodic Fibonacci word.” The following definition is quite old in the literature but is rarely given a specific name [4].

Definition 2.5 *For* $n \geq 1$*, let* $t_{(a,b,n)}$ *be* $f_{(a,b,n)}$ *with the last two symbols interchanged. That is, if* $f_{(a,b,n)} = p_{(a,b,n)}xy$*, then* $t_{(a,b,n)} = p_{(a,b,n)}yx$*.*

The following follows immediately from Proposition 2.4.3 and Definition 2.5.

Corollary 2.6 *Let* $n \geq 3$*. Then*

$$f_{(a,b,n-1)}f_{(a,b,n-2)} = f_{(a,b,n-2)}t_{(a,b,n-1)}$$
$$f_{(a,b,n-1)}t_{(a,b,n-2)} = f_{(a,b,n-2)}f_{(a,b,n-1)}.$$

A key realization in the formation of this paper was that the natural self-similar structure was one that included overlapping cells. To that end it will be useful to express how copies of f and t overlap with each other efficiently.

Lemma 2.7 *If $n \geq 3$, then $f_{(a,b,n)}$ ends in $f^{r-2}_{(a,b,n-1)} f_{(a,b,n-2)}$. Similarly, $t_{(a,b,n)}$ begins with $f^{r-2}_{(a,b,n-1)} f_{(a,b,n-2)}$.*

Proof. We decompose $f_{(a,b,n)}$ and $t_{(a,b,n)}$ according to the rules allowed by Definition 2.2 and using the notational shortcut of Definition 2.1:

$$f_{(a,b,n)} = f^2_{(a,b,n-1)} f^{r-2}_{(a,b,n-1)} f_{(a,b,n-2)}$$

and

$$\begin{aligned} t_{(a,b,n)} &= f^r_{(a,b,n-1)} t_{(a,b,n-2)} \\ &= f^{r-2}_{(a,b,n-1)} f^2_{(a,b,n-1)} t_{(a,b,n-2)} \\ &= f^{r-2}_{(a,b,n-1)} f_{(a,b,n-2)} f^{s-1}_{(a,b,n-2)} f_{(a,b,n-3)} f_{(a,b,n-1)} t_{(a,b,n-2)}. \end{aligned}$$

□

By virtue of Lemma 2.7, we can consistently define a new word by allowing $f_{(a,b,n)}$ and $t_{(a,b,n)}$ to "overlap" across the shared affix mentioned in the lemma. Inspired by roughly overlapping the letters "f" and "t", we call this new word $I_{(a,b,n)}$. Visually, we can depict $I_{(a,b,n)}$ as follows, where the square brackets show where $f_{(a,b,n)}$ is located while the curly brackets show $t_{(a,b,n)}$.

$$I_{(a,b,n)} = \left[f^2_{(a,b,n-1)} \left\{ f^{r-2}_{(a,b,n-1)} f_{(a,b,n-2)} \right] f^{s-1}_{(a,b,n-2)} f_{(a,b,n-3)} f_{(a,b,n-1)} t_{(a,b,n-2)} \right\}.$$

Definition 2.8 *For $n \geq 3$, define the word $I_{(a,b,n)}$, the "overlapping concatenation," by*

$$I_{(a,b,n)} = f^2_{(a,b,n-1)} f^{r-2}_{(a,b,n-1)} f_{(a,b,n-2)} f^{s-1}_{(a,b,n-2)} f_{(a,b,n-3)} f_{(a,b,n-1)} t_{(a,b,n-2)}.$$

It will be convenient to have multiple identities for how the words f, t, and I combine. We begin with a simple corollary, which is clear from Definition 2.8

Corollary 2.9 *For all $n \geq 3$, we have $I_{(a,b,n)} = f^2_{(a,b,n-1)} t_{(a,b,n)}$.*

The next proposition is the first where we make use of Corollary 2.6. It will be an important tool is recognizing a self-similar cell structure in $f_{(a,b,n)}$.

Proposition 2.10 *Let $n \geq 4$. Then $f^2_{(a,b,n)} = f^r_{(a,b,n-1)} I_{(a,b,n-1)} t^{r-1}_{(a,b,n-1)}$.*

Proof. We begin with the left hand side and begin with a decomposition via Definition 2.2. We then use Corollary 2.6 to repeatedly swap $f_{(a,b,n-1)} f_{(a,b,n-2)}$ with $f_{(a,b,n-2)} t_{(a,b,n-1)}$. Finally Corollary 2.9 is used to regather terms.

$$\begin{aligned} f^2_{(a,b,n)} &= f_{(a,b,n)} f_{(a,b,n)} \\ &= f^r_{(a,b,n-1)} f_{(a,b,n-2)} f^r_{(a,b,n-1)} f_{(a,b,n-2)} \\ &= f^r_{(a,b,n-1)} f_{(a,b,n-2)} f_{(a,b,n-2)} t^r_{(a,b,n-1)} \\ &= f^r_{(a,b,n-1)} I_{(a,b,n-1)} t^{r-1}_{(a,b,n-1)} \end{aligned}$$

□

3 Decompositions of Biperiodic Fibonacci Words

We now have a collection of biperiodic Fibonacci words f, t, I. It will be possible to represent $f_{(a,b,n)}$ as a composition of the words $f_{(a,b,n-l)}$, $t_{(a,b,n-l)}$, and $I_{(a,b,n-l)}$, where l is the periodicity of the substitution morphism of [10]. That is, the period of the self-similar structure follows from the substitution rules, but the contribution here is the *arrangement and number* of the cells in the self-similar structure.

The following sequence of lemmas and the theorem they culminate in depend on the parity of a, b, and n lining up in particular ways. Specifically, the eight ways to choose parity of a, b, and n break down into just three cases.

Lemma	a	b	n	r	s
3.1	even	even	either	even	even
3.1	even	odd	even	even	odd
3.1	odd	even	odd	even	odd
3.3	odd	even	even	odd	even
3.3	even	odd	odd	odd	even
3.2	odd	odd	either	odd	odd

The lower limits on n are chosen to accommodate applications of Proposition 2.10 as it is invoked for words with $n-1$, $n-2$, and $n-3$ indices. As a reminder, we assume that $a, b \geq 2$ here and leave the necessary modifications for when one or the other is 1 to Section 4.

Lemma 3.1 *Suppose $n \geq 5$ and r is even; then $f_{(a,b,n)}$ is composed of copies of $f_{(a,b,n-2)}$ and $t_{(a,b,n-2)}$ according to the formula*

$$f_{(a,b,n)} = \left(f^{s}_{(a,b,n-2)} I_{(a,b,n-2)} t^{s-1}_{(a,b,n-2)}\right)^{r/2} f_{(a,b,n-2)}.$$

Proof. Since $n \geq 5$, $n-1 \geq 4$, so using Proposition 2.10 on $f^2_{(a,b,n-1)}$ we get:

$$\begin{aligned}\left(f^{s}_{(a,b,n-2)} I_{(a,b,n-2)} t^{s-1}_{(a,b,n-2)}\right)^{r/2} f_{(a,b,n-2)} &= \left(f^2_{(a,b,n-1)}\right)^{r/2} f_{(a,b,n-2)} \\ &= f^{r}_{(a,b,n-1)} f_{(a,b,n-2)} \\ &= f_{(a,b,n)}.\end{aligned}$$

□

Lemma 3.2 *When both r and s are odd and $n \geq 6$, then*

$$f_{(a,b,n)} = \left[\left(f^2_{(a,b,n-2)}\right)^{(s+1)/2} t_{(a,b,n-3)} \left(f^2_{(a,b,n-2)}\right)^{(s-1)/2} f_{(a,b,n-3)}\right]^{(r-1)/2} \left(f^2_{(a,b,n-2)}\right)^{(s+1)/2} t_{(a,b,n-3)} \quad (1)$$

Note that subscripts do not indicate a non-trivial self-similar structure since $n-2 \neq n-3$, but recall that by Proposition 2.10 we know that $f^2_{(a,b,n-2)}$ splits into pieces of $f_{(a,b,n-3)}$ and $t_{(a,b,n-3)}$. Still, however, unless $a=b=k$ this is also not self-similar: Recall from Definition 1.1 that a cell structure of period 3 must give the same arrangement of cells for $f_{(a,b,n)}$, $f_{(a,b,n+3)}$, $f_{(a,b,n+6)}$... but the above decomposition does not satisfy this, since n and $n+3$ have opposite parities, so when applying Lemma 3.2 to $f_{(a,b,n+3)}$, one must reverse the roles of r and s. Nevertheless, the cell structure is consistent between $f_{(a,b,n)}$ and $f_{(a,b,n+6)}$, as n and $n+6$ have the same parity. So to really have a self-similar structure this lemma needs to be invoked again on each of its constituent words to get a period of 6, see Theorem 3.4. Invoking this lemma twice does give the similarity between $f_{(a,b,n)}$ and $f_{(a,b,n-6)}$ that is indicated by the substitution morphism in [10][Theorem 20 (2)].

Proof. Similar to the previous proof consider the right hand side of (1) and simplify it to

$$\left[f^{s+1}_{(a,b,n-2)} t_{(a,b,n-3)} f^{s-1}_{(a,b,n-2)} f_{(a,b,n-3)}\right]^{(r-1)/2} f^{s+1}_{(a,b,n-2)} t_{(a,b,n-3)}$$

Using Corollary 2.6 and Definition 2.2 we have the replacement:

$$f^{s+1}_{(a,b,n-2)} t_{(a,b,n-3)} = f^{s}_{(a,b,n-2)} f_{(a,b,n-3)} f_{(a,b,n-2)}$$

In light of which we have

$$\begin{aligned}
\text{RHS of (1)} &= \left[f^{s}_{(a,b,n-2)} f_{(a,b,n-3)} f^{s}_{(a,b,n-2)} f_{(a,b,n-3)}\right]^{(r-1)/2} f^{s}_{(a,b,n-2)} f_{(a,b,n-3)} f_{(a,b,n-2)} \\
&= \left[f_{(a,b,n-1)} f_{(a,b,n-1)}\right]^{(r-1)/2} f_{(a,b,n-1)} f_{(a,b,n-2)} \\
&= f^{r-1}_{(a,b,n-1)} f_{(a,b,n-1)} f_{(a,b,n-2)} \\
&= f_{(a,b,n)}
\end{aligned}$$

We applied Proposition 2.10 to $f_{(a,b,n-2)}$ so $n-2 \geq 4$ must be assumed. ☐

Lemma 3.3 *When r is odd and s is even and $n \geq 7$,*

$$\begin{aligned}
f_{(a,b,n)} = \Bigg\{ &\left[\left(f^2_{(a,b,n-3)}\right)^{(r+1)/2} t_{(a,b,n-4)} \left(f^2_{(a,b,n-3)}\right)^{(r-1)/2} f_{(a,b,n-4)}\right]^{s/2} \\
&\left(f^2_{(a,b,n-3)}\right)^{(r+1)/2} f_{(a,b,n-4)} \\
&\left[\left(f^2_{(a,b,n-3)}\right)^{(r+1)/2} t_{(a,b,n-4)} \left(f^2_{(a,b,n-3)}\right)^{(r-1)/2} f_{(a,b,n-4)}\right]^{(s-2)/2} \\
&\left(f^2_{(a,b,n-3)}\right)^{(r+1)/2} f_{(a,b,n-4)} \Bigg\}^{(r-1)/2} \\
&\left[\left(f^2_{(a,b,n-3)}\right)^{(r+1)/2} t_{(a,b,n-4)} \left(f^2_{(a,b,n-3)}\right)^{(r-1)/2} f_{(a,b,n-4)}\right]^{s/2} \\
&\left(f^2_{(a,b,n-3)}\right)^{(r+1)/2} f_{(a,b,n-4)}
\end{aligned} \tag{2}$$

Again, Proposition 2.10 lets us obtain a self-similar structure, this time at level $n-4$. This similarity agrees with [10, Theorem 20] for even n cases and also works with a and b exchanging roles for odd n cases. There are obviously no corresponding $k-$Fibonacci formulas since this case does not appear when $a=b$.

Proof. The strategy is the same, first we simplify the exponents in (2) and use Corollary 2.6 to perform the exchange

$$f^{r+1}_{(a,b,n-3)}t_{(a,b,n-4)} = f^{r}_{(a,b,n-3)}f_{(a,b,n-4)}f_{(a,b,n-3)}$$

After repeated uses of this fact, the simplification of exponents, and Definition 2.2 we arrive at

$$\begin{aligned}\text{RHS of (2)} &= \Big\{ \left[f^2_{(a,b,n-2)}\right]^{s/2} f^{r+1}_{(a,b,n-3)}f_{(a,b,n-4)} \\ &= \left[f^2_{(a,b,n-2)}\right]^{(s-2)/2} f^{r+1}_{(a,b,n-3)}t_{(a,b,n-4)}\Big\}^{(r-1)/2} \\ &= \left[f^2_{(a,b,n-2)}\right]^{s/2} f^{r+1}_{(a,b,n-3)}f_{(a,b,n-4)}.\end{aligned}$$

As s is even this can be further simplified to:

$$\left\{ f^{s}_{(a,b,n-2)}f^{r+1}_{(a,b,n-3)}f_{(a,b,n-4)}f^{s-2}_{(a,b,n-2)}f^{r+1}_{(a,b,n-3)}t_{(a,b,n-4)} \right\}^{(r-1)/2} \\ f^{s}_{(a,b,n-2)}f^{r+1}_{(a,b,n-3)}f_{(a,b,n-4)}.$$

Then using Definition 2.2 and Corollary 2.6 the $n-4$ terms can be combined and ultimately reduce to $f_{(a,b,n)}$.

We applied Proposition 2.10 to $f_{(a,b,n-3)}$ so $n-3\geq 4$ must be assumed.

□

The following theorem follows from Lemmas 3.1 through 3.3 and several uses of Proposition 2.10. The lemmas contain the technical details but are not written to make clear what the self-similar structure of a biperiodic Fibonacci word is. The following theorem is a restatement of the lemmas making it clear that $f_{(a,b,n)}$ is an overlapping concatenation of similar words.

Theorem 3.4 *The biperiodic Fibonacci words $f_{(a,b,n)}$ have a self-similar cell structure in the sense of Definition 1.1 with the period l determined by the parities of a and b. The self-similar cell structures are shown in Table 1. Note: if a and b are both odd the displayed cell structure structure must be applied twice to be truly self-similar. The minimal n that these structures hold for are the same as in Lemmas 3.1 through 3.3.*

- *If a and b are both even, then r is even so $l=2$ and the "r is even" cell structure describes $f_{(a,b,n)}$ for all n.*
- *If a is even and b is odd and n is even then $r=b$ is odd while $s=a$ is even so $l=4$ and the "r is odd, s is even" cell structure describes $f_{(a,b,n)}$.*

$f_{(a,b,n)} =$	r is even $\left(f^{s}_{(a,b,n-2)} I_{(a,b,n-2)} t^{s-1}_{(a,b,n-2)}\right)^{r/2} f_{(a,b,n-2)}$
$f_{(a,b,n)} =$	r, s are odd $\left[\left(f^{r}_{(a,b,n-3)} I_{(a,b,n-3)} t^{r-1}_{(a,b,n-3)}\right)^{(s+1)/2} t_{(a,b,n-3)} \left(f^{r}_{(a,b,n-3)} I_{(a,b,n-3)} t^{r-1}_{(a,b,n-3)}\right)^{(s-1)/2} f_{(a,b,n-3)}\right]^{(r-1)/2} \left(f^{r}_{(a,b,n-3)} I_{(a,b,n-3)} t^{r-1}_{(a,b,n-3)}\right)^{(s+1)/2} t_{(a,b,n-3)}$
$f_{(a,b,n)} =$	r is odd and s is even $\left\{\left[\left(f^{s}_{(a,b,n-4)} I_{(a,b,n-4)} t^{s-1}_{(a,b,n-4)}\right)^{(r+1)/2} t_{(a,b,n-4)} \left(f^{s}_{(a,b,n-4)} I_{(a,b,n-4)} t^{s-1}_{(a,b,n-4)}\right)^{(r-1)/2} f_{(a,b,n-4)}\right]^{s/2} \left(f^{s}_{(a,b,n-4)} I_{(a,b,n-4)} t^{s-1}_{(a,b,n-4)}\right)^{(r+1)/2} f_{(a,b,n-4)} \left[\left(f^{s}_{(a,b,n-4)} I_{(a,b,n-4)} t^{s-1}_{(a,b,n-4)}\right)^{(r+1)/2} t_{(a,b,n-4)} \left(f^{s}_{(a,b,n-4)} I_{(a,b,n-4)} t^{s-1}_{(a,b,n-4)}\right)^{(r-1)/2} f_{(a,b,n-4)}\right]^{(s-2)/2} \left(f^{s}_{(a,b,n-4)} I_{(a,b,n-4)} t^{s-1}_{(a,b,n-4)}\right)^{(r+1)/2} t_{(a,b,n-4)}\right\}^{(r-1)/2} \left[\left(f^{s}_{(a,b,n-4)} I_{(a,b,n-4)} t^{s-1}_{(a,b,n-4)}\right)^{(r+1)/2} t_{(a,b,n-4)} \left(f^{s}_{(a,b,n-4)} I_{(a,b,n-4)} t^{s-1}_{(a,b,n-4)}\right)^{(r-1)/2} f_{(a,b,n-4)}\right]^{s/2} \left(f^{s}_{(a,b,n-4)} I_{(a,b,n-4)} t^{s-1}_{(a,b,n-4)}\right)^{(r+1)/2} f_{(a,b,n-4)}$

Table 1: The self-similar structures of for $f_{(a,b,n)}$. The first is for when r is even. The second for when r and s are both odd, however in this case the self-similar structure requires that this decomposition be performed twice. The third is for when r is odd but s is even.

- *If a is even and b is odd and n is odd then $r = a$ is even so $l = 2$ and the "r is even" cell structure describes $f_{(a,b,n)}$.*

- *If a is odd and b is even and n is even then $r = b$ is even so $l = 2$ and the "r is even" cell structure describes $f_{(a,b,n)}$.*

- *If a is odd and b is even and n is odd then $r = a$ is odd and $s = b$ is even so $l = 4$ and the "r is odd, s is even" cell structure describes $f_{(a,b,n)}$.*

- *If a and b are odd then $l = 6$ and the "r, s are odd" case is used twice to break $f_{(a,b,n)}$ into copies of $f_{(a,b,n-3)}$ and then into copies of $f_{(a,b,n-6)}$, the result is a self-similar cell structure that describes $f_{(a,b,n)}$.*

The proof of the theorem is just using in each case the relevant choice of lemma from this section.

Recall that the original motivation for seeing the self-similar structure of $f_{(a,b,n)}$ was to understand the geometry of fractals which are produced from the words by means of a scaling limit construction. In forthcoming works by the same authors the self-similar decompositions of $f_{(a,b,n)}$ shown in Table 1 will be used to identify iterated functions systems that produce the same fractals. The benefit of this will be to ease the analysis of the geometry of such fractals as the theory of iterated functions systems is extensive. In particular by providing an iterated function system it becomes possible to state when the fractal associated to a biperiodic Fibonacci word is homeomorphic to a line and when it is not using only a finite number of $f_{(a,b,n)}$'s.

Corollary 3.5 *Let $a = b$. Then if $k = a = b$ is even we have $l = 2$ and the "r and s are even" cell structure describes $f_{(a,b,n)}$. If $k = a = b$ is odd then $l = 3$ and the "r and s are odd" cell structure describes $f_{(a,b,n)}$*

Proof. For the even case this is just Theorem 3.4. However, for the odd case one can just apply Lemma 3.2 once to supply a self-similar cell structure. The concern that a and b switch roles between $f_{(a,b,n)}$ and $f_{(a,b,n+3)}$ does not apply if we have $a = b$. Thus $l = 3$ and the "r and s are odd" self-similar cell structure describes $f_{(k,k,n)}$. □

4 When a or b is 1

When a or b is equal to 1 some modifications of the foregoing need to be made. The necessity of the modifications is clearly demonstrated by attempting to write $t_{(1,b,1)}$. Since $f_{(1,b,1)} = 1$ is it difficult to know how to interpret $t_{(1,b,1)}$ by transposing the last *two* symbols in $f_{(1,b,1)}$. Any result in this paper that involved the use of $t_{(a,b,n)}$ will need the minimal value of n assumed increased by 1 to accommodate $t_{(1,b,1)}$ not being defined. For the sake of brevity we do not provide any of the modified proofs for the $a = 1$ or $b = 1$ cases but simply indicate what the new hypotheses must be. The proofs use the same methods and techniques and the details can quickly be provided by any motivated reader.

The other modification that needs to be made is that Definition 2.8 does not make sense when $r = 1$. So instead Corollary 2.9 can be proven using the following alternative definition of $I_{(a,b,n)}$ when $r = 1$.

Definition 4.1 *For $n \geq 5$ and $r = 1$ let $I_{(a,b,n)}$ consist of the words $f_{(a,b,n)}$, $f_{(a,b,n)}$, and $t_{(a,b,n)}$ in order where each adjacent pair of words overlaps by a copy of $f_{(a,b,n-2)}$.*

With this alternate definition of I Theorem 3.4 holds with the same conclusion in the case where $r = 1$.

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