Mixed Finite Element Method Based on the Crank–Nicolson Scheme for Darcy Flows in Porous Media

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Abstract - This article is devoted to the a priori error estimates of the fully discrete Crank–Nicolson approximation for the Darcy flows. Optimal order error estimates in both $L^\infty(0,T;H^1(\Omega))$ and $L^\infty(0,T;L^2(\Omega))$ norms are established for the lowest order. Numerical experiments confirm the theoretical analysis regarding convergence rates.

Keywords : mixed finite element method; Crank–Nicolson scheme; error estimates; optimal order

Mathematics Subject Classification (2020) **:** 65M12; 65M15; 65M60; 35Q35; 76S05; 65N15; 65N30; 35K15; 35K05.

1 Introduction

The paper is dedicated to the analysis of mixed finite element approximations of the solutions of the system of equations modeling the flow of compressible fluids in porous media subject to the Darcy law. This phenomenon generated a lot of interest in the research community such as engineering, environmental and groundwater hydrology and in medicine.

Darcy's law is commonly related to viscous fluid laminar flows in porous media and is characterized by the permeability coefficient, which is obtained empirically in order to match the linear relation between the velocity vector and the pressure gradient. Darcy's equation has also been obtained rigorously within the context of homogenization and other averaging/upscaling techniques [25, 30]. From a hydrodynamic point of view, Darcy's equation is interpreted as the momentum equation. Darcy's equation, the continuity equation, and the equation of state serve as the framework to model processes in reservoirs [9, 23]. For a slightly compressible fluid, the original PDE system reduces to a scalar linear second order parabolic equation for the density only.

The popular numerical method for modeling flow in porous media is the mixed finite element approximations (e.g., [10,13,21,27,28]). This method is widely used because of its inherent conservation properties and because it produces accurate flux even for highly homogeneous media with large jumps in the conductivity (permeability) tensor [12]. Since the pioneering work of Raviart and Thomas [32], the method has become a standard way of deriving high order conservative approximations. We recommend to the reader [6] for general accounts of the mixed method. Douglas et al. [11] proposed semidiscrete mixed finite element methods to approximate the solution of the system (3.11a) – (3.11b) and obtained optimal order error estimates for the pressure in L^2 under reasonable assumptions. In [28], one of the authors has further analyzed the method and obtained optimal order error estimates for the flux variable in the several norms of interest.

There exist several time-discretization methods to deal with the parabolic equations such as the backward Euler method, the Crank–Nicolson method and the Runge–Kutta method [17]. As is known to all, the Crank–Nicolson scheme [8] was first proposed by Crank and Nicolson for the heat-conduction equation in 1947, and it is unconditionally stable with second-order accuracy. Moreover, because of its high accuracy and unconditional stability, the scheme has been widely used in many PDEs. So we use the Crank–Nicolson scheme and prove the optimal order of convergence.

The paper is organized as follows. In Section 2, we introduce notation and some of relevant results. We present a semi discrete mixed finite element approximation for the problem. Existence and uniqueness are discussed and some known results are recalled. In Section 4, we derive error estimates for the two relevant functions. We consider the fully discrete mixed finite element method based on the Crank–Nicolson scheme to approximate the solution of the system (3.2). The optimal order estimates are established in L^2-norms for density and momentum under reasonable assumptions on the regularity of solutions. In Section 5, the results of a few numerical experiments using the lowest Raviart-Thomas mixed finite element in the two-dimensions are reported. These results support our theoretical analysis regarding convergence rates.

2 Preliminaries and Auxiliaries

We consider a fluid in a porous medium occupying a bounded domain $\Omega \subset \mathbb{R}^d$, $d \geq 2$ with boundary $\partial\Omega$. Let $\mathbf{x} \in \mathbb{R}^d$, $0 < T < \infty$ and $t \in (0, T]$ be the spatial and time variables respectively. The fluid flow has velocity $\mathbf{v}(\mathbf{x}, t) \in \mathbb{R}^d$, pressure $p(\mathbf{x}, t) \in \mathbb{R}$, density $\rho(\mathbf{x}, t) \in \mathbb{R}_+ = [0, \infty)$ and dynamic viscosity μ and permeability $\kappa > 0$.

The Darcy equation, which is considered as a momentum equation, is studied in [4, 29] and has the form

$$\mathbf{v}(\mathbf{x}, t) = -\frac{\kappa}{\mu}\nabla p(\mathbf{x}, t). \tag{2.1}$$

This relationship describes the linear relationship between the velocity $\mathbf{v}$ of the creep flow and the gradient of the pressure p, which is valid when the velocity $\mathbf{v}$ is extremely small [3] observed in oil and water wells due to low permeability. A theoretical derivation of Darcy's law can be found in [16, 24, 34].

Multiplying both sides of the equation (2.1) to ρ, we find that

$$\rho(\mathbf{x}, t)\mathbf{v}(\mathbf{x}, t) = -\frac{\kappa}{\mu}\rho(\mathbf{x}, t)\nabla p(\mathbf{x}, t). \tag{2.2}$$

We recall that the fluid's compressibility for isothermal conditions is

$$\varpi = -\frac{1}{V}\frac{dV}{dp} = \frac{1}{\rho}\frac{d\rho}{dp},$$

where V, here, denotes the fluid's volume. In many cases such as (isothermal) compressible liquids, ϖ is assumed to be a constant [4, 23]. In particular, it is a small positive constant for (isothermal) slightly compressible fluids such as crude oil and water. This condition is commonly used in petroleum and reservoir engineering [1, 9], where the fluid dynamics in porous media have important applications. The current paper is focused on (isothermal) slightly compressible fluids, hence, we study the following equation of state

$$\frac{1}{\rho}\frac{d\rho}{dp} = \varpi, \quad \text{where the constant compressibility } \varpi > 0 \text{ is small.} \tag{2.3}$$

Hence

$$\nabla\rho = \varpi\rho\nabla p, \quad \text{or} \quad \rho\nabla p = \varpi^{-1}\nabla\rho. \tag{2.4}$$

Combining (2.2) and (2.4) implies that

$$\rho(\mathbf{x},t)\mathbf{v}(\mathbf{x},t) = -\frac{\kappa}{\mu\varpi}\nabla\rho(\mathbf{x},t). \tag{2.5}$$

The continuity equation is

$$\phi\rho_t(\mathbf{x},t) + \nabla\cdot(\rho(\mathbf{x},t)\mathbf{v}(\mathbf{x},t)) = f(\mathbf{x},t), \tag{2.6}$$

where $\phi \in (0;1)$ is the constant porosity, and f is the external mass flow rate.

By combining (2.5) and (2.6) we have

$$\begin{aligned} &\mathbf{m}(\mathbf{x},t) + \beta\nabla\rho(\mathbf{x},t) = 0,\\ &\phi\rho_t(\mathbf{x},t) + \nabla\cdot\mathbf{m}(\mathbf{x},t) = f(\mathbf{x},t), \end{aligned}$$

where $\mathbf{m}(\mathbf{x},t) = \rho(\mathbf{x},t)\mathbf{v}(\mathbf{x},t)$, $\beta = \frac{\kappa}{\mu\varpi} > 0$.

By rescaling the variables $\rho \to \beta\rho$, $\phi \to \beta^{-1}\phi$, we can assume $\beta = 1$ to obtain the system of equations

$$\begin{aligned} &\mathbf{m}(\mathbf{x},t) + \nabla\rho(\mathbf{x},t) = 0,\\ &\phi\rho_t(\mathbf{x},t) + \nabla\cdot\mathbf{m}(\mathbf{x},t) = f(\mathbf{x},t). \end{aligned} \tag{2.7}$$

We recall some elementary inequalities that will be used in this paper.

For all $a, b \geq 0$,

$$2^{-1}(a^p + b^p) \leq (a+b)^p \leq 2^{|p-1|}(a^p + b^p) \quad \text{for all } p > 0. \tag{2.8}$$

Lemma 2.1 (Young's inequality, general version). *Let $N \in \mathbb{N}$, $p_i \in [1,\infty)$, i=1,..., N, be such that $\sum_{i=1}^{N} \frac{1}{p_i} = 1$, and let $c_i > 0$ be such that $\prod_{i=1}^{N} c_i = 1$. Then*

$$\prod_{i=1}^{N} a_i \leq \sum_{i=1}^{N} \frac{c_i^{p_i}}{p_i} a_i^{p_i}$$

for all non-negative real numbers $a_i, i = 1,\ldots,N$.

We recall a discrete version of Gronwall Lemma in backward difference form, which is useful later. It can be proven without much difficulty by following the ideas of the proof in Gronwall Lemma.

Lemma 2.2 *Assume $\ell \geq 0, 1-\ell\tau > 0$ and the nonnegative sequences $\{a_n\}_{n=0}^{\infty}$, $\{g_n\}_{n=0}^{\infty}$ satisfying*

$$\frac{a_n - a_{n-1}}{\tau} - \ell a_n \leq g_n, \quad n = 1,2,3\ldots$$

then

$$a_n \leq (1-\ell\tau)^{-n}\Big(a_0 + \tau\sum_{i=1}^{n}(1-\ell\tau)^{i-1}g_i\Big). \tag{2.9}$$

Proof. Let $\bar{a}_n = (1-\ell\tau)^n a_n$. A simple calculation shows that

$$\frac{\bar{a}_n - \bar{a}_{n-1}}{\tau} = (1-\ell\tau)^{n-1}\Big(\frac{a_n - a_{n-1}}{\tau} - \ell a_n\Big) \le (1-\ell\tau)^{n-1} g_n.$$

Summation over n leads to

$$\frac{\bar{a}_n - \bar{a}_0}{\tau} \le \sum_{i=1}^{n} (1-\ell\tau)^{i-1} g_i,$$

and hence (2.9) holds true. □

Notations. Throughout this paper, we assume that Ω is an open, bounded subset of $\mathbb{R}^d$, with $d = 2, 3, \ldots$, and has C^1-boundary $\partial\Omega$. For $s \in [1,\infty)$, we denote $L^s(\Omega)$ be the set of s-integrable functions on Ω and $(L^s(\Omega))^d$ the space of d-dimensional vectors which have all components in $L^s(\Omega)$. We denote $(\cdot,\cdot)$ the inner product in either $L^s(\Omega)$ or $(L^s(\Omega))^d$ that is $(\xi,\eta) = \int_\Omega \xi\eta dx$ or $(\boldsymbol{\xi},\boldsymbol{\eta}) = \int_\Omega \boldsymbol{\xi}\cdot\boldsymbol{\eta} dx$ and $\|u\|_{L^s(\Omega)} = \left(\int_\Omega |u(x)|^s dx\right)^{1/s}$ for standard Lebesgue norm of the measurable function. The notation $\langle\cdot,\cdot\rangle$ will be used for the $L^2(\partial\Omega)$ inner-product. For $m \ge 0, s \in [1,\infty]$, we denote the Sobolev spaces by $W^{m,s}(\Omega) = \{v \in L^s(\Omega), : D^\alpha v \in L^s(\Omega), |\alpha| \le m\}$ and the norm of $W^{m,s}(\Omega)$ by $\|v\|_{W^{m,s}(\Omega)} = \left(\sum_{|\alpha|\le m} \int_\Omega |D^\alpha u|^s dx\right)^{1/s}$, and $\|v\|_{W^{m,\infty}(\Omega)} = \sum_{|\alpha|\le m} \operatorname{ess\,sup}_\Omega |D^\alpha u|$. Finally we define $L^s(0,T;X)$ to be the space of all measurable functions $v : [0,T] \to X$ with the norm $\|v\|_{L^s(0,T;X)} = \left(\int^T \|v(t)\|_X^s \, dt\right)^{1/s}$, and $L^\infty(0,T;X)$ to be the space of all measurable functions $v : [0,T] \to X$ such that $v : t \to \|v(t)\|_X$ is essentially bounded on $[0,T]$ with the norm $\|v\|_{L^\infty(0,T;X)} = \operatorname{ess\,sup}_{t\in[0,T]} \|v(t)\|_X$.

Throughout this paper, we use short hand notations, $\|\cdot\|_k = \|\cdot\|_{k,2}$ and $\|\cdot\| = \|\cdot\|_{0,2}$ and $\|\cdot\|_{r,p} = \|\cdot\|_{W^{r,p}}$ $\|\rho(t)\| = \|\rho(\cdot,t)\|_{L^2(\Omega)}, \forall t \ge 0$ and $\rho^0(\cdot) = \rho(\cdot,0)$.

Throughout this paper, we use $C, C_1, C_2, \ldots$ to denote a generic positive constant whose value may change from place to place but are independent of the parameters of the discretization.

3 The Mixed Finite Element Method

We study the initial– boundary value problem or IBVP

$$\mathbf{m}(\mathbf{x},t) + \nabla\rho(\mathbf{x},t) = 0 \quad (\mathbf{x},t) \in \Omega \times (0,T), \tag{3.1a}$$

$$\phi\rho_t(\mathbf{x},t) + \nabla\cdot\mathbf{m}(\mathbf{x},t) = f(\mathbf{x},t) \quad (\mathbf{x},t) \in \Omega \times (0,T). \tag{3.1b}$$

The initial and boundary conditions:

$$\rho(\mathbf{x},0) = \rho_0(\mathbf{x}) \text{ in } \Omega, \quad \rho(\mathbf{x},t) = \psi(\mathbf{x},t) \text{ on } \partial\Omega \times (0,T),$$

we also require at $t = 0$: $\rho_0(\mathbf{x}) = \psi(\mathbf{x},0)$ on boundary $\partial\Omega$. The mixed formulation of (3.1a)–(3.1b) reads as follows. Find $(\mathbf{m},\rho) : [0,T] \to H(\mathrm{div},\Omega) \times \mathrm{L}^2(\Omega) \equiv \mathcal{M} \times \mathcal{R}$ such that

$$(\mathbf{m},\mathbf{v}) - (\rho,\nabla\cdot\mathbf{v}) = -\langle\psi,\mathbf{v}\cdot\nu\rangle \quad \forall \mathbf{v} \in \mathcal{M}, \tag{3.2a}$$

$$\phi\,(\rho_t,q) + (\nabla\cdot\mathbf{m},q) = (f,q) \quad \forall q \in \mathcal{R}, \tag{3.2b}$$

where $\langle\cdot,\cdot\rangle$ is the inner product in $L^2(\partial\Omega)$ and ν denotes the unit outer normal vector to $\partial\Omega$.

Let $\{\mathcal{T}_h\}_h$ be a quasi-regular polygonalization of Ω (by triangles, rectangles, tetrahedron or possibly hexahedron), with $\max_{\tau\in\mathcal{T}_h} \operatorname{diam} \tau \leq$ h. The discrete subspace $\mathcal{M}_h \times \mathcal{R}_h \subset \mathcal{M} \times \mathcal{R}$ is defined as

$$\mathcal{M}_h = \{\mathbf{v} \in H(\operatorname{div}, \Omega) | \mathbf{v} = \mathbf{a} + b\mathbf{x} \text{ for all } T \in \mathcal{T}_h\},$$
$$\mathcal{R}_h = \{q \in L^2(\Omega) | q \text{ is constant on each element } T \in \mathcal{T}_h\}.$$

So $\mathcal{M}_h$ denotes the RT_0 space (the Raviart-Thomas-Nedelec [7,32]) and $\mathcal{R}_h$ is the space of piecewise constant functions.

For momentum, let $\Pi : \mathcal{M} \to \mathcal{M}_h$ be the Raviart-Thomas projection [31], which satisfies

$$(\nabla \cdot (\Pi\mathbf{m} - \mathbf{m}), q) = 0, \quad \text{for all } \mathbf{m} \in \mathcal{M}, q \in \mathcal{R}_h. \tag{3.3}$$

For density, we use the standard L^2-projection operator, see in [7], $\pi : \mathcal{R} \to \mathcal{R}_h$, satisfying

$$\begin{aligned} (\pi\rho - \rho, q) &= 0, \quad \text{for all } \rho \in \mathcal{R}, q \in \mathcal{R}_h, \\ (\pi\rho - \rho, \nabla \cdot \mathbf{m}_h) &= 0, \quad \text{for all } \mathbf{m}_h \in \mathcal{M}_h, \rho \in \mathcal{R}. \end{aligned} \tag{3.4}$$

This projection has well-known approximation properties, e.g. [5,6,18].

$$\|\Pi\mathbf{m}\| \leq C\big(\|\mathbf{m}\| + h \|\nabla \cdot \mathbf{m}\| \big), \quad \forall \mathbf{m} \in \mathcal{M} \cap (W^{1,2}(\Omega))^d. \tag{3.5}$$
$$\|\Pi\mathbf{m} - \mathbf{m}\| \leq Ch \|\mathbf{m}\|_1, \quad \forall \mathbf{m} \in \mathcal{M} \cap (W^{1,2}(\Omega))^d. \tag{3.6}$$
$$\|\pi\rho\| \leq C \|\rho\|, \quad \forall \rho \in L^2(\Omega). \tag{3.7}$$
$$\|\pi\rho - \rho\| + h \|\pi\rho - \rho\|_1 \leq Ch^2 \|\rho\|_2, \quad \forall \rho \in W^{2,2}(\Omega). \tag{3.8}$$

The two projections π and Π preserve the commuting property $\operatorname{div} \circ \Pi = \pi \circ \operatorname{div} : V \to \mathcal{R}_h$.

We shall also find useful the following inequalities valid for each $\mathcal{T} \in \mathcal{T}_h$

$$\|\nabla \cdot \mathbf{m}\|_{L^2(\mathcal{T})} \leq Ch^{-1} \|\mathbf{m}\|_{L^2(\mathcal{T})}, \quad \mathbf{m} \in \mathcal{M}_h. \tag{3.9}$$
$$\|\mathbf{m} \cdot \nu\|_{L^2(\partial\mathcal{T})} \leq Ch^{-\frac{1}{2}} \|\mathbf{m}\|_{L^2(\mathcal{T})}, \quad \mathbf{m} \in \mathcal{M}_h. \tag{3.10}$$

The mixed finite element problem is stated as: Find a pair $(\mathbf{m}_h, \rho_h) : [0, T] \to \mathcal{M}_h \times \mathcal{R}_h$ such that

$$(\mathbf{m}_h, \mathbf{v}) - (\rho_h, \nabla \cdot \mathbf{v}) = -\langle \psi, \mathbf{v} \cdot \nu \rangle \quad \forall \mathbf{v} \in \mathcal{M}_h, \tag{3.11a}$$
$$\phi\,(\rho_{ht}, q) + (\nabla \cdot \mathbf{m}_h, q) = (f, q) \quad \forall q \in \mathcal{R}_h. \tag{3.11b}$$

Initially we take $\rho_h^0 = \pi\rho_0$. With this choice, we obtain for all $q \in \mathcal{R}_h$

$$\left(\rho_h^0, q\right) = (\pi\rho_0(\mathbf{x}), q)\,.$$

We assume that $f \in L^2(0, T; L^2(\Omega))$, $\psi \in L^2(0, T; L^2(\partial\Omega))$ and $\rho_0 \in L^2(\Omega)$. Then, see for instance [15,19,20,33], pages 156–158, for more details, there exists a unique weak solution for (3.2a)–(3.2b) in the following sense: there exists a function $\rho \in L^2(0, T; H_0^1(\Omega)) \cap C(0, T; L^2(\Omega))$.

4 Fully Discrete Problem Based on the Crank–Nicolson Scheme

The discretization scheme we want to consider is implicit and it is based on the use of the Crank–Nicolson method as discretization in time and on the use of the finite element mesh described above.

We first divide the interval $[0,T]$ into N equally-spaced subintervals by the following points

$$0 = t_0 < t_1 < \ldots < t_{N+1} = T$$

with $t_i = i\tau$, $t_{i-\frac{1}{2}} = (i-\frac{1}{2})\tau$, for time step $\tau = T/N$. For a smooth function φ on $[0,T]$, we define $\varphi^i = \varphi(\cdot, t_i)$ and $\varphi^{i-\frac{1}{2}} = \varphi(\cdot, t_{i-\frac{1}{2}})$. We shall denote by $\bar{\varphi}^i$ the following arithmetic mean value, when $\{\varphi^i\}_{i=0}^{N+1}$ is a discrete function, between the two time levels $i-1$ and i:

$$\bar{\varphi}^i = \frac{\varphi^i + \varphi^{i-1}}{2}.$$

We also define

$$\delta\varphi^i = \frac{\varphi^i - \varphi^{i-1}}{\tau}, \quad \text{and} \quad \delta^2\varphi^{i+1} = \delta(\delta\varphi^{i+1}) = \frac{\varphi^{i+1} - 2\varphi^i + \varphi^{i-1}}{\tau^2} \quad \forall i = 1, 2, \ldots, N+1.$$

The fully discrete time mixed finite element approximation to (3.2) is defined as follows: Given $\{f^i\}_{i=1}^{N+1} \in L^2(\Omega)$, $\{\psi^i\}_{i=1}^{N+1} \in L^2(\partial\Omega)$. Find a pair $(\mathbf{m}_h^i, \rho_h^i)$ in $\mathcal{M}_h \times \mathcal{R}_h$, $i = 1, 2, \ldots, N+1$ such that

$$\left(\bar{\mathbf{m}}_h^i, \mathbf{v}\right) - \left(\bar{\rho}_h^i, \nabla \cdot \mathbf{v}\right) = -\left\langle \bar{\psi}^i, \mathbf{v} \cdot \nu \right\rangle, \quad \forall \mathbf{v} \in \mathcal{M}_h, \tag{4.1a}$$

$$\phi\left(\delta\rho_h^i, q\right) + \left(\nabla \cdot \bar{\mathbf{m}}_h^i, q\right) = \left(\bar{f}^i, q\right), \quad \forall q \in \mathcal{R}_h. \tag{4.1b}$$

Initially, we take $(\mathbf{m}_h^0, \rho_h^0) = (\pi\nabla\rho_0, \pi\rho_0)$. With this choice we obtain

$$\left(\mathbf{m}_h^0, \mathbf{v}\right) - \left(\pi\rho^0, \nabla \cdot \mathbf{v}\right) = -\left\langle \psi^0, \mathbf{v} \cdot \nu \right\rangle, \quad \forall \mathbf{v} \in \mathcal{M}_h, \tag{4.2}$$

$$\left(\rho_h^0, q\right) = (\pi\rho_0, q), \quad \forall q \in \mathcal{R}_h. \tag{4.3}$$

Remark 4.1 We can use $f^{i-\frac{1}{2}}$ in place of $\bar{f}^i$ in the second equation of (4.1). If f is a continuous function of time then we define $f^i = f(\cdot, t_i)$. If f is less regular, then we define

$$f^i = \frac{1}{\tau}\int_{t_{i-1}}^{t_i} f(\cdot, t)dt, \quad i = 1, \ldots N+1.$$

Lemma 4.2 (Stability) *Let $(\mathbf{m}_h^n, \rho_h^n)$ solve the fully discrete finite element approximation (4.1) for each time step n, $n = 1, \ldots, N+1$. Suppose that $(\mathbf{m}, \rho) \in \mathcal{M} \times \mathcal{R}$, $f \in L^\infty(0,T; L^2(\Omega))$, and $\psi \in L^\infty(0,T; L^2(\partial\Omega))$, $\rho_0 \in L^2(\Omega)$. Then, there exists a positive constant C independent of τ such that for τ sufficiently small*

(i) For all $n = 1, 2, \ldots, N+1$,

$$\|\rho_h^n\|^2 + \tau\sum_{i=1}^{n}\left\|\bar{\mathbf{m}}_h^i\right\|^2 \le C(h)\Big(\|\rho_0\|^2 + \sum_{i=1}^{n}\tau\left\|\bar{f}^i\right\|^2 + \left\|\bar{\psi}^i\right\|_{L^2(\partial\Omega)}^2\Big). \tag{4.4}$$

(ii) For all $n = 1, 2, \ldots, N+1$,

$$\|\mathbf{m}_h^n\|^2 \le C(h)\Big(\|\rho_0\|^2 + \left\|\psi^0\right\|_{L^2(\partial\Omega)}^2 + \sum_{i=1}^{n}\sum_{j=1}^{i}\tau\left\|\bar{f}^j\right\|^2 + \left\|\bar{\psi}^j\right\|_{L^2(\partial\Omega)}^2\Big). \tag{4.5}$$

Proof.

(i) Let $(q, \mathbf{v}) = (\bar{\rho}_h^i, \bar{\mathbf{m}}_h^i)$ in (4.1a) and (4.1b). Adding the resultant equations gives

$$\left(\bar{\mathbf{m}}_h^i, \bar{\mathbf{m}}_h^i\right) + \phi\left(\delta\rho_h^i, \bar{\rho}_h^i\right) = \left(\bar{f}^i, \bar{\rho}_h^i\right) - \left\langle \bar{\psi}^i, \bar{\mathbf{m}}_h^i \cdot \nu \right\rangle .$$

Using Hölder's inequality, the triangle inequality and the inverse inequality (3.10), we find that

$$\begin{aligned}\left\|\bar{\mathbf{m}}_h^i\right\|^2 + \frac{\phi}{2\tau}\left(\left\|\rho_h^i\right\|^2 - \left\|\rho_h^{i-1}\right\|^2\right) &\leq \left\|\bar{f}^i\right\|\left\|\bar{\rho}_h^i\right\| + \left\|\bar{\psi}^i\right\|_{L^2(\partial\Omega)}\left\|\bar{\mathbf{m}}_h^i\right\|_{L^2(\partial\Omega)} \\ &\leq \frac{1}{2}\left\|\bar{f}^i\right\|\left(\left\|\rho_h^i\right\| + \left\|\rho_h^{i-1}\right\|\right)\left\|\bar{\rho}_h^i\right\| + Ch^{-\frac{1}{2}}\left\|\bar{\psi}^i\right\|_{L^2(\partial\Omega)}\left\|\bar{\mathbf{m}}_h^i\right\| .\end{aligned}$$

It follows from the Young inequality that

$$\left\|\bar{\mathbf{m}}_h^i\right\|^2 + \phi(1+\tau)\frac{\left\|\rho_h^i\right\|^2 - \left\|\rho_h^{i-1}\right\|^2}{\tau} - 2\phi\left\|\rho_h^i\right\|^2 \leq 2\phi^{-1}\left\|\bar{f}^i\right\|^2 + Ch^{-1}\left\|\bar{\psi}^i\right\|^2_{L^2(\partial\Omega)} .$$

By the discrete Gronwall inequality (2.9) for $n = 1, 2, \ldots, N+1$,

$$\begin{aligned}\|\rho_h^n\|^2 + \frac{\tau}{\phi(1+\tau)}\sum_{i=1}^n \left\|\bar{\mathbf{m}}_h^i\right\|^2 &\leq C\Big(\frac{1-\tau}{1+\tau}\Big)^{-n}\left(\left\|\rho_h^0\right\|^2 + \sum_{i=1}^n \tau\left\|\bar{f}^i\right\|^2 + h^{-1}\left\|\bar{\psi}^i\right\|^2_{L^2(\partial\Omega)} \right) \\ &\leq C(h)e^{n\tau}\Big(\left\|\pi\rho^0\right\|^2 + \sum_{i=1}^n \tau\left\|\bar{f}^i\right\|^2 + \left\|\bar{\psi}^i\right\|^2_{L^2(\partial\Omega)}\Big) \\ &\leq C(h)\Big(\|\rho_0\|^2 + \sum_{i=1}^n \tau\left\|\bar{f}^i\right\|^2 + \left\|\bar{\psi}^i\right\|^2_{L^2(\partial\Omega)}\Big).\end{aligned}$$

We completed the proof of (4.4).

(ii) By taking $\mathbf{v} = 2(\mathbf{m}_h^i - \mathbf{m}_h^{i-1})$ in the Eq. (4.1a) we find that

$$\left\|\mathbf{m}_h^i\right\|^2 - \left\|\mathbf{m}_h^{i-1}\right\|^2 = 2\left(\bar{\rho}_h^i, \nabla\cdot(\mathbf{m}_h^i - \mathbf{m}_h^{i-1})\right) - 2\left\langle \bar{\psi}^i, (\mathbf{m}_h^i - \mathbf{m}_h^{i-1})\cdot\nu \right\rangle .$$

Using Hölder's inequality, the triangle inequality and the inverse inequality (3.9)-(3.10), we find that

$$\begin{aligned}\left\|\mathbf{m}_h^i\right\|^2 - \left\|\mathbf{m}_h^{i-1}\right\|^2 &\leq \left\|\rho_h^i + \rho_h^{i-1}\right\|\left\|\nabla\cdot(\mathbf{m}_h^i - \mathbf{m}_h^{i-1})\right\| + 2\left\|\bar{\psi}^i\right\|_{L^2(\partial\Omega)}\left\|\mathbf{m}_h^i - \mathbf{m}_h^{i-1}\right\|_{L^2(\partial\Omega)} \\ &\leq Ch^{-1}\left\|\rho_h^i + \rho_h^{i-1}\right\|\left(\left\|\mathbf{m}_h^i\right\| + \left\|\mathbf{m}_h^{i-1}\right\|\right) + Ch^{-\frac{1}{2}}\left\|\bar{\psi}^i\right\|_{L^2(\partial\Omega)}\left(\left\|\mathbf{m}_h^i\right\| + \left\|\mathbf{m}_h^{i-1}\right\|\right) .\end{aligned}$$

Applying Cauchy- Schwartz and (2.8) gives

$$\frac{3}{2}(\left\|\mathbf{m}_h^i\right\|^2 - \left\|\mathbf{m}_h^{i-1}\right\|^2) - \left\|\mathbf{m}_h^i\right\|^2 \leq C(h)\left(\left\|\rho_h^i\right\|^2 + \left\|\rho_h^{i-1}\right\|^2 + \left\|\bar{\psi}^i\right\|^2_{L^2(\partial\Omega)}\right). \quad (4.6)$$

Inserting (4.4) into (4.6) leads to

$$(\left\|\mathbf{m}_h^i\right\|^2 - \left\|\mathbf{m}_h^{i-1}\right\|^2) - \frac{2}{3}\left\|\mathbf{m}_h^i\right\|^2 \leq C(h)\Big(\|\rho_0\|^2 + \sum_{j=1}^i \tau\left\|\bar{f}^j\right\|^2 + \left\|\bar{\psi}^j\right\|^2_{L^2(\partial\Omega)}\Big). \quad (4.7)$$

Due to the discrete Gronwall's inequality (2.9) with $\tau = 1$ and $\ell = \frac{2}{3}$, we find that

$$\|\mathbf{m}_h^n\|^2 \leq C \left\|\mathbf{m}_h^0\right\|^2 + C(h) \sum_{i=1}^{n} \big(\|\rho_0\|^2 + \sum_{j=1}^{i} \tau \left\|\bar{f}^j\right\|^2 + \left\|\bar{\psi}^j\right\|_{L^2(\partial\Omega)}^2 \big). \tag{4.8}$$

Since $\left(\mathbf{m}_h^0, \mathbf{v}\right) - \left(\pi\rho^0, \nabla \cdot \mathbf{v}\right) = -\left\langle \psi^0, \mathbf{v} \cdot \nu \right\rangle$, let $\mathbf{v} = \mathbf{m}_h^0$ then

$$\begin{aligned} \left\|\mathbf{m}_h^0\right\|^2 &\leq \|\pi\rho_0\| \left\|\nabla \cdot \mathbf{m}_h^0\right\| + \left\|\psi^0\right\|_{L^2(\partial\Omega)} \left\|\mathbf{m}_h^0\right\|_{L^2(\partial\Omega)} \\ &\leq C(h)(\|\pi\rho_0\| + \left\|\psi^0\right\|_{L^2(\partial\Omega)}) \left\|\mathbf{m}_h^0\right\|. \end{aligned} \tag{4.9}$$

Combining (4.8) and (4.9) yields

$$\|\mathbf{m}_h^n\|^2 \leq C(h) \big(\|\rho_0\|^2 + \left\|\psi^0\right\|_{L^2(\partial\Omega)}^2 + \sum_{i=1}^{n} \sum_{j=1}^{i} \tau \left\|\bar{f}^j\right\|^2 + \left\|\bar{\psi}^j\right\|_{L^2(\partial\Omega)}^2 \big).$$

The proof is complete. □

Lemma 4.3 *Let $(\mathbf{m}_h^n, \rho_h^n)$ solve the fully discrete finite element approximation (4.1) for each time step n, $n = 2, \ldots, N+1$. Suppose that $f \in L^\infty(0, T; L^2(\Omega))$, $\psi \in L^\infty(0, T; L^2(\partial\Omega))$, $\rho_0 \in L^2(\Omega)$. Then, there exists a positive constant C independent of τ such that for τ sufficiently small*

$$\left\| \frac{\rho_h^n - \rho_h^{n-1}}{\tau} \right\| \leq C(h) \Big(\|\rho_0\|^2 + \left\|\psi^0\right\|_{L^2(\partial\Omega)}^2 + \left\|\bar{f}^1\right\|^2 + \left\|\bar{\psi}^1\right\|_{L^2(\partial\Omega)}^2 + \sum_{i=2}^{n} \left\|\delta \bar{f}^i\right\| + \left\|\delta\bar{\psi}^i\right\| \Big). \tag{4.10}$$

Proof. Acting the discrete operator δ on (4.1a) and (4.1b) we get, for all $i = 1 \ldots, N+1$

$$\left(\delta\bar{\mathbf{m}}_h^i, \mathbf{v}\right) - \left(\delta\bar{\rho}_h^i, \nabla \cdot \mathbf{v}\right) = -\left\langle \delta\bar{\psi}^i, \mathbf{v} \cdot \nu \right\rangle, \quad \forall \mathbf{v} \in \mathcal{M}_h, \tag{4.11a}$$

$$\phi \left(\delta^2 \rho_h^i, q\right) + \left(\delta\nabla \cdot \bar{\mathbf{m}}_h^i, q\right) = \left(\delta\bar{f}^i, q\right), \quad \forall q \in \mathcal{R}_h. \tag{4.11b}$$

Taking $\mathbf{v} = \delta\bar{\mathbf{m}}_h^i$ in (4.11a) and $q = \delta\bar{\rho}_h^i$ in (4.11b), adding the resultant equations we obtain

$$\phi \frac{\left\|\delta\rho_h^i\right\|^2 - \left\|\delta\rho_h^{i-1}\right\|^2}{2\tau} + \left\|\delta\bar{\mathbf{m}}_h^i\right\|^2 = \left(\delta\bar{f}^i, \delta\bar{\rho}_h^i\right) + \left\langle \delta\bar{\psi}^i, \delta\bar{\mathbf{m}}_h^i \cdot \nu \right\rangle.$$

Thanks to the use of Hölder's inequality, triangle inequality and inverse inequality (3.10), we obtain that

$$\begin{aligned} \phi \frac{\left\|\delta\rho_h^i\right\|^2 - \left\|\delta\rho_h^{i-1}\right\|^2}{2\tau} + \left\|\delta\bar{\mathbf{m}}_h^i\right\|^2 &\leq \left\|\delta\bar{f}^i\right\| \left\|\delta\bar{\rho}_h^i\right\| + \left\|\delta\bar{\psi}^i\right\|_{L^2(\partial\Omega)} \left\|\delta\bar{\mathbf{m}}_h^i\right\|_{L^2(\partial\Omega)} \\ &\leq \left\|\delta\bar{f}^i\right\| \left\|\delta\bar{\rho}_h^i\right\| + Ch^{-\frac{1}{2}} \left\|\delta\bar{\psi}^i\right\|_{L^2(\partial\Omega)} \left\|\delta\bar{\mathbf{m}}_h^i\right\|. \end{aligned}$$

It follows from Young's inequality that

$$\phi(1+\tau) \frac{\left\|\delta\rho_h^i\right\|^2 - \left\|\delta\rho_h^{i-1}\right\|^2}{\tau} - 2\phi \left\|\delta\rho_h^i\right\|^2 + \left\|\delta\bar{\mathbf{m}}_h^i\right\|^2 \leq 2\phi^{-1} \left\|\delta\bar{f}^i\right\|^2 + Ch^{-1} \left\|\delta\bar{\psi}^i\right\|_{L^2(\partial\Omega)}^2,$$

By discrete Gronwall's inequality (2.9)

$$\|\delta\rho_h^n\|^2 \leq C\left\|\delta\rho_h^1\right\|^2 + C(h)\sum_{i=2}^{n}\tau\left\|\delta\bar{f}^i\right\|^2 + \left\|\delta\bar{\psi}^i\right\|_{L^2(\partial\Omega)}^2. \tag{4.12}$$

Let us estimate $\left\|\delta\rho_h^1\right\|^2$. At the step $i=1$, taking $q=\delta\rho_h^1$ in Eq.(4.1b) we find that

$$\phi\left(\delta\rho_h^1,\delta\rho_h^1\right) + \left(\nabla\cdot\bar{\mathbf{m}}_h^1,\delta\rho_h^1\right) = \left(\bar{f}^1,\delta\rho_h^1\right).$$

Thus

$$\begin{aligned}\phi\left\|\delta\rho_h^1\right\|^2 &\leq C\big(\left\|\nabla\cdot\bar{\mathbf{m}}_h^1\right\|^2 + \left\|\bar{f}^1\right\|^2\big) \leq Ch^{-2}\left\|\bar{\mathbf{m}}_h^1\right\|^2 + C\left\|\bar{f}^1\right\|^2 \\ &\leq C(h)(\left\|\mathbf{m}_h^1\right\|^2 + \left\|\mathbf{m}_h^0\right\|)^2 + \left\|\bar{f}^1\right\|^2).\end{aligned} \tag{4.13}$$

By (4.13) and (4.5), it implies that

$$\phi\left\|\delta\rho_h^1\right\|^2 \leq C(h)\big(\left\|\rho_0\right\|^2 + \left\|\psi^0\right\|_{L^2(\partial\Omega)}^2 + (\tau+1)\left\|\bar{f}^1\right\|^2 + \left\|\bar{\psi}^1\right\|_{L^2(\partial\Omega)}^2\big). \tag{4.14}$$

Substituting (4.14) into (4.12) shows (4.10) holds true.
The proof is complete. □

4.1 Error Analysis for the Fully Discrete Method

In this section we derive an error estimate for the fully discrete scheme. First, we give some results that are crucial in getting the convergence results.

Lemma 4.4 *For $n\geq 1$ if $\rho_{tt},\rho_{ttt}\in L^2(0,T;L^2(\Omega))$, then*

$$(i)\quad \left\|\bar{\rho}^n - \rho^{n-\frac{1}{2}}\right\|^2 \leq C\tau^3\int_{t_{n-1}}^{t_n}\|\rho_{tt}\|^2\,dt. \tag{4.15}$$

$$(ii)\quad \left\|\delta\rho^n - \rho_t^{n-\frac{1}{2}}\right\|^2 \leq C\tau^3\int_{t_{n-1}}^{t_n}\|\rho_{ttt}\|^2\,dt. \tag{4.16}$$

Proof. (i) By Taylor expansion with integral remainder

$$\rho^n = \rho^{n-\frac{1}{2}} + \frac{\tau}{2}\rho_t^{n-\frac{1}{2}} + \int_{t_{n-\frac{1}{2}}}^{t_n}\rho_{tt}(t)(t_n-t)dt.$$

$$\rho^{n-1} = \rho^{n-\frac{1}{2}} - \frac{\tau}{2}\rho_t^{n-\frac{1}{2}} + \int_{t_{n-\frac{1}{2}}}^{t_{n-1}}\rho_{tt}(t)(t-t_{n-1})dt.$$

This implies that

$$\begin{aligned}\left\|\frac{\rho^n+\rho^{n-1}}{2} - \rho^{n-\frac{1}{2}}\right\|^2 &= \frac{1}{2}\int_\Omega\Big|\int_{t_{n-\frac{1}{2}}}^{t_n}\rho_{tt}(t)(t_n-t)dt + \int_{t_{n-\frac{1}{2}}}^{t_{n-1}}\rho_{tt}(t)(t-t_{n-1})dt\Big|^2dx \\ &\leq \int_\Omega\Big(\int_{t_{n-\frac{1}{2}}}^{t_n}\rho_{tt}(t)(t_n-t)dt\Big)^2 + \Big(\int_{t_{n-\frac{1}{2}}}^{t_{n-1}}\rho_{tt}(t)(t-t_{n-1})dt\Big)^2dx.\end{aligned} \tag{4.17}$$

We estimate the right hand side by Hölder's inequality

$$
\begin{aligned}
&\int_\Omega \Big(\int_{t_{n-\frac{1}{2}}}^{t_n} \rho_{tt}(t)(t_n-t)dt\Big)^2 + \Big(\int_{t_{n-\frac{1}{2}}}^{t_{n-1}} \rho_{tt}(t)(t-t_{n-1})dt\Big)^2 dx \\
&\quad \le \int_\Omega \Big(\int_{t_{n-\frac{1}{2}}}^{t_n} |\rho_{tt}|^2 dt \int_{t_{n-\frac{1}{2}}}^{t_n} (t_n-t)^2 dt\Big) dx + \int_\Omega \Big(\int_{t_{n-1}}^{t_{n-\frac{1}{2}}} |\rho_{tt}|^2 dt \int_{t_{n-1}}^{t_{n-\frac{1}{2}}} (t_n-t)^2 dt\Big) dx \qquad (4.18)\\
&\quad \le \frac{\tau^3}{24}\Big(\int_\Omega \int_{t_{n-\frac{1}{2}}}^{t_n} |\rho_{tt}|^2 dt dx + \int_\Omega \int_{t_{n-1}}^{t_{n-\frac{1}{2}}} |\rho_{tt}|^2 dt dx\Big) \le \frac{\tau^3}{12}\int_{t_{n-1}}^{t_n} \|\rho_{tt}\|^2 dt.
\end{aligned}
$$

Then (4.15) follows directly from inserting (4.18) into (4.17).
(ii) Similar proof for (4.16). By Taylor expansion with integral remainder

$$
\rho^n = \rho^{n-\frac{1}{2}} + \frac{\tau}{2!}\rho_t^{n-\frac{1}{2}} + \frac{\tau^2}{3!}\rho_{tt}^{n-\frac{1}{2}} + \frac{1}{3!}\int_{t_{n-\frac{1}{2}}}^{t_n} \rho_{ttt}(t)(t_n-t)^2 dt.
$$

$$
\rho^{n-1} = \rho^{n-\frac{1}{2}} - \frac{\tau}{2!}\rho_t^{n-\frac{1}{2}} + \frac{\tau^2}{3!}\rho_{tt}^{n-\frac{1}{2}} - \frac{1}{3!}\int_{t_{n-\frac{1}{2}}}^{t_{n-1}} \rho_{ttt}(t)(t-t_{n-1})^2 dt.
$$

Using (2.8) and Hölder's inequality shows that

$$
\begin{aligned}
&\Big\|\frac{\rho^n-\rho^{n-1}}{\tau} - \rho_t^{n-\frac{1}{2}}\Big\|^2 = \frac{\tau^{-2}}{36}\int_\Omega \Big|\int_{t_{n-\frac{1}{2}}}^{t_n} \rho_{ttt}(t)(t_n-t)^2 dt + \int_{t_{n-\frac{1}{2}}}^{t_{n-1}} \rho_{ttt}(t)(t-t_{n-1})^2 dt\Big|^2 dx \\
&\quad \le \frac{\tau^{-2}}{36}\int_\Omega \Big(\int_{t_{n-\frac{1}{2}}}^{t_n} \rho_{ttt}(t)(t_n-t)^2 dt\Big)^2 + \Big(\int_{t_{n-\frac{1}{2}}}^{t_{n-1}} \rho_{ttt}(t)(t-t_{n-1})^2 dt\Big)^2 dx \\
&\quad \le \frac{1}{36}\tau^{-2}\int_\Omega \int_{t_{n-\frac{1}{2}}}^{t_n} |\rho_{ttt}|^2 dt \int_{t_{n-\frac{1}{2}}}^{t_n} (t_n-t)^4 dt dx + \int_\Omega \int_{t_{n-1}}^{t_{n-\frac{1}{2}}} |\rho_{ttt}|^2 dt \int_{t_{n-1}}^{t_{n-\frac{1}{2}}} (t_n-t)^4 dt dx \\
&\quad \le \frac{\tau^3}{5760}\Big(\int_\Omega \int_{t_{n-\frac{1}{2}}}^{t_n} |\rho_{ttt}|^2 dt dx + \int_\Omega \int_{t_{n-1}}^{t_{n-\frac{1}{2}}} |\rho_{ttt}|^2 dt dx\Big) \le \frac{\tau^3}{2880}\int_{t_{n-1}}^{t_n} \|\rho_{ttt}\|^2 dt.
\end{aligned}
$$

Then (4.16) follows. □

Lemma 4.5 *For $n \ge 2$, suppose $\rho_{tt}, \rho_{ttt} \in L^2(0,T;L^2(\Omega))$. Then there is a positive constant C such that*

$$
\Big\|\delta(\bar{\rho}^n - \rho^{n-\frac{1}{2}})\Big\|^2 \le C\tau \int_{t_{n-2}}^{t_n} \|\rho_{tt}\|^2 dt. \qquad (4.19a)
$$

$$
\Big\|\delta(\delta\rho^n - \rho_t^{n-\frac{1}{2}})\Big\|^2 \le C\tau \int_{t_{n-2}}^{t_n} \|\rho_{ttt}\|^2 dt. \qquad (4.19b)
$$

Proof. Each estimate is a result of using the Taylor theorem with integral remainder and the Hölder inequality. □

First, we derive an error estimate.

Subtract (3.2a) from (4.1a) to obtain

$$\left(\bar{\mathbf{m}}^i - \bar{\mathbf{m}}_h^i, \mathbf{v}\right) - \left(\bar{\rho}^i - \bar{\rho}_h^i, \nabla\cdot\mathbf{v}\right) = 0. \tag{4.20}$$

From (3.2b) we have

$$\phi\left(\bar{\rho}_t^i, q\right) + \left(\nabla\cdot\bar{\mathbf{m}}^i, q\right) = \left(\bar{f}^i, q\right). \tag{4.21}$$

Subtract (4.1b) from (4.21) to obtain

$$\phi\left(\delta(\rho^i - \rho_h^i), q\right) + \left(\nabla\cdot(\bar{\mathbf{m}}^i - \bar{\mathbf{m}}_h^i), q\right) = \phi\left(\delta\rho^i - \rho_t^{i-\frac{1}{2}}, q\right) - \phi\left(\bar{\rho}_t^i - \rho_t^{i-\frac{1}{2}}, q\right). \tag{4.22}$$

Then from (4.20) and (4.22) we have

$$\left(\bar{\mathbf{m}}^i - \bar{\mathbf{m}}_h^i, \mathbf{v}\right) - \left(\bar{\rho}^i - \bar{\rho}_h^i, \nabla\cdot\mathbf{v}\right) = 0, \forall\mathbf{v}\in\mathcal{M}_h, \tag{4.23a}$$

$$\phi\left(\delta(\rho^i - \rho_h^i), q\right) + \left(\nabla\cdot(\bar{\mathbf{m}}^i - \bar{\mathbf{m}}_h^i), q\right) = \phi\left(\delta\rho^i - \rho_t^{i-\frac{1}{2}}, q\right) - \phi\left(\bar{\rho}_t^i - \rho_t^{i-\frac{1}{2}}, q\right), \forall q\in\mathcal{R}_h \tag{4.23b}$$

Let

$$\rho^i - \rho_h^i = \rho^i - \pi\rho^i + \pi\rho^i - \rho_h^i = \zeta^i + \theta^i = \chi^i.$$
$$\mathbf{m}^i - \mathbf{m}_h^i = \mathbf{m}^i - \Pi\mathbf{m}^i + \Pi\mathbf{m}^i - \mathbf{m}_h^i = \xi^i + \vartheta^i = \eta^i.$$

From (3.8) and (3.6) we have

$$\left\|\zeta^i\right\| = \left\|\rho^i - \pi\rho^i\right\| \le Ch^2\left\|\rho^i\right\|_2 \le Ch^2\Big(\left\|\rho^0\right\|_2 + \int_0^{t_i}\|\rho_t\|_2\,dt\Big). \tag{4.24}$$

$$\left\|\xi^i\right\| = \left\|\mathbf{m}^i - \Pi\mathbf{m}^i\right\| \le Ch\left\|\mathbf{m}^i\right\|_1 \le Ch\Big(\left\|\mathbf{m}^0\right\|_1 + \int_0^{t_i}\|\mathbf{m}_t\|_1\,dt\Big). \tag{4.25}$$

Theorem 4.6 *Let $(\mathbf{m}^n, \rho^n)$ solve problem (3.2a)–(3.2b) and $(\mathbf{m}_h^n, \rho_h^n)$ solve the fully discrete finite element approximation (4.1a)–(4.1b) for each time step n, $n = 1, \ldots, N+1$. Suppose that $(\mathbf{m}^0, \rho^0) \in (W^{1,2}(\Omega))^d, W^{2,2}(\Omega))$, $(\mathbf{m}_t, \rho_t) \in (L^1(0,T;W^{1,2}(\Omega))^d, L^1(0,T;W^{2,2}(\Omega)))$, and $\rho_{tt}, \rho_{ttt} \in L^2(0,T;L^2(\Omega))$. Then, there exists a positive constant C independent of h and τ such that, for τ sufficiently small,*

$$(i)\quad \|\rho^n - \rho_h^n\| \le C\tau^2\Big(\int_0^{t_n}\|\rho_{tt}\|^2 + \|\rho_{ttt}\|^2\,dt\Big)^{\frac{1}{2}} + Ch^2\Big(\left\|\rho^0\right\|_2 + \int_0^{t_n}\|\rho_t\|_2\,dt\Big). \tag{4.26}$$

$$(ii)\quad \|\mathbf{m}^n - \mathbf{m}_h^n\| \le C\tau^2\Big(\int_0^{t_n}\|\rho_{tt}\|^2 + \|\rho_{ttt}\|^2\,dt\Big)^{\frac{1}{2}} + Ch\Big(\left\|\mathbf{m}^0\right\|_1 + \int_0^{t_n}\|\mathbf{m}_t\|_1\,dt\Big). \tag{4.27}$$

Proof.

(i) For any $q\in\mathcal{R}_h, \mathbf{v}\in\mathcal{M}_h$, then from (4.23a) and (4.23b), recalling the projectors in (3.4) and (3.3), we end up with

$$\left(\bar{\vartheta}^i, \mathbf{v}\right) - \left(\bar{\theta}^i, \nabla\cdot\mathbf{v}\right) = 0, \quad \forall\mathbf{v}\in\mathcal{M}_h, \tag{4.28a}$$

$$\phi\left(\delta\theta^i, q\right) + \left(\nabla\cdot\bar{\vartheta}^i, q\right) = \phi\left(\delta\rho^i - \rho_t^{i-\frac{1}{2}}, q\right) - \phi\left(\bar{\rho}_t^i - \rho_t^{i-\frac{1}{2}}, q\right), \quad \forall q\in\mathcal{R}_h. \tag{4.28b}$$

Now choosing $(q, \mathbf{v}) = (\bar{\theta}^i, \bar{\vartheta}^i)$ in (4.28a) and (4.28b), and adding the resulting equations we obtain

$$\phi\left(\delta\theta^i, \bar{\theta}^i\right) + \left\|\bar{\vartheta}^i\right\|^2 = \phi\left(\delta\rho^i - \rho_t^{i-\frac{1}{2}}, \bar{\theta}^i\right) - \phi\left(\bar{\rho}_t^i - \rho_t^{i-\frac{1}{2}}, \bar{\theta}^i\right).$$

Applying the Young inequality and (2.8) yields

$$\begin{aligned}\phi\frac{\left\|\theta^i\right\|^2 - \left\|\theta^{i-1}\right\|^2}{2\tau} + \left\|\bar{\vartheta}^i\right\|^2 &\leq \frac{\phi}{2\varepsilon}\left(\left\|\delta\rho^i - \rho_t^{i-\frac{1}{2}}\right\|^2 + \left\|\bar{\rho}_t^i - \rho_t^{i-\frac{1}{2}}\right\|^2\right) + \varepsilon\phi\left\|\bar{\theta}^i\right\|^2 \\ &\leq \frac{\phi}{2\varepsilon}\left(\left\|\delta\rho^i - \rho_t^{i-\frac{1}{2}}\right\|^2 + \left\|\bar{\rho}_t^i - \rho_t^{i-\frac{1}{2}}\right\|^2\right) + \frac{\phi\varepsilon}{2}\Big(\left\|\theta^i\right\|^2 + \left\|\theta^{i-1}\right\|^2\Big),\end{aligned}$$

it follows that

$$\left\|\theta^i\right\|^2 \leq \frac{1+\tau\varepsilon}{1-\tau\varepsilon}\left\|\theta^{i-1}\right\|^2 + \frac{\tau}{\varepsilon(1-\tau\varepsilon)}\left(\left\|\delta\rho^i - \rho_t^{i-\frac{1}{2}}\right\|^2 + \left\|\bar{\rho}_t^i - \rho_t^{i-\frac{1}{2}}\right\|^2\right).$$

Choosing $\varepsilon > 0$ such that $1 - \tau\varepsilon > 1/2$, we find that

$$\left\|\theta^i\right\|^2 \leq C\left(\left\|\theta^{i-1}\right\|^2 + \tau\left(\left\|\delta\rho^i - \rho_t^{i-\frac{1}{2}}\right\|^2 + \left\|\bar{\rho}_t^i - \rho_t^{i-\frac{1}{2}}\right\|^2\right)\right).$$

According to Lemma 4.4, we have

$$\left\|\delta\rho^i - \rho_t^{i-\frac{1}{2}}\right\|^2 + \left\|\bar{\rho}_t^i - \rho_t^{i-\frac{1}{2}}\right\|^2 \leq C\tau^3\int_{t_{i-1}}^{t_i}\|\rho_{tt}\|^2 + \|\rho_{ttt}\|^2\,dt.$$

We obtain

$$\left\|\theta^i\right\|^2 \leq C\left(\left\|\theta^{i-1}\right\|^2 + \tau^4\int_{t_{i-1}}^{t_i}\|\rho_{tt}\|^2 + \|\rho_{ttt}\|^2\,dt\right).$$

Noting that $\theta^0 = 0$ and adding all equations for $i = 1, 2, \ldots n \leq N$, we get

$$\|\theta^n\|^2 \leq C\tau^4\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\|\rho_{tt}\|^2 + \|\rho_{ttt}\|^2\,dt. \tag{4.29}$$

The result (4.26) follows straightforwardly using (4.29), (4.24) and the triangle inequality.

(ii) For any $q \in \mathcal{R}_h, \mathbf{v} \in \mathcal{M}_h$, from (4.20) and (4.22), using L^2-project and elliptic projection, we find that

$$\left(\delta\bar{\vartheta}^i, \mathbf{v}\right) - \left(\delta\bar{\theta}^i, \nabla\cdot\mathbf{v}\right) = 0, \quad \forall\mathbf{v} \in \mathcal{M}_h, \tag{4.30a}$$

$$\phi\left(\delta\theta^i, q\right) + \left(\nabla\cdot\bar{\vartheta}^i, q\right) = \phi\left(\delta\rho^i - \rho_t^{i-\frac{1}{2}}, q\right) - \phi\left(\bar{\rho}_t^i - \rho_t^{i-\frac{1}{2}}, q\right), \quad \forall q \in \mathcal{R}_h. \tag{4.30b}$$

From the sum of Eq. (4.30b) with $q = \delta\bar{\theta}^i$ and Eq. (4.30a) with $\mathbf{v} = \bar{\vartheta}^i$, applying Young's inequality, we obtain

$$\phi\left\|\delta\bar{\theta}^i\right\|^2 + \frac{1}{2\tau}\left(\left\|\vartheta^i\right\|^2 - \left\|\vartheta^{i-1}\right\|^2\right) \leq \phi\left\|\delta\rho^i - \rho_t^{i-\frac{1}{2}}\right\|^2 + \phi\left\|\bar{\rho}_t^i - \rho_t^{i-\frac{1}{2}}\right\|^2 + \frac{\phi}{2}\left\|\delta\bar{\theta}^i\right\|^2.$$

By Lemma 4.4, (4.15)-(4.16) we find that

$$\frac{\left\|\vartheta^i\right\|^2-\left\|\vartheta^{i-1}\right\|^2}{\tau} \leq C\tau^3 \int_{t_{i-1}}^{t_i} \|\rho_{tt}\|^2+\|\rho_{ttt}\|^2\,dt.$$

The discrete Gronwall lemma (Lemma 2.2 with $\ell=0$) yields

$$\|\vartheta^n\|^2 \leq C\left\|\vartheta^0\right\|^2+C\tau^4\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\|\rho_{tt}\|^2+\|\rho_{ttt}\|^2\,dt, \tag{4.31}$$

since

$$\left(\mathbf{m}^0-\mathbf{m}_h^0,\mathbf{v}\right)+\left(\rho^0-\pi\rho^0,\nabla\cdot\mathbf{v}\right)=\left(\Pi\mathbf{m}^0-\mathbf{m}^0,\mathbf{v}\right)\quad \forall \mathbf{v}\in\mathcal{M}_h.$$

Let $\mathbf{v}=\Pi\mathbf{m}^0-\mathbf{m}_h^0$, then

$$\left\|\Pi\mathbf{m}^0-\mathbf{m}_h^0\right\| \leq \left\|\Pi\mathbf{m}^0-\mathbf{m}^0\right\| \leq Ch^1\left\|\mathbf{m}^0\right\|_1. \tag{4.32}$$

Inserting (4.32) into (4.31) we find that

$$\|\vartheta^n\| \leq Ch\left\|\mathbf{m}^0\right\|_1+C\tau^2\Big(\int_0^{t_n}\|\rho_{tt}\|^2+\|\rho_{ttt}\|^2\,dt\Big)^{\frac{1}{2}}. \tag{4.33}$$

Combining (4.33), (4.25) and the triangle inequality we complete the proof (4.27). □

Theorem 4.7 *Let $(\mathbf{m}^n,\rho^n)$ solve problem (3.2) and $(\mathbf{m}_h^n,\rho_h^n)$ solve the fully discrete finite element approximation (4.1) for each time step n, $n=1,\ldots,N+1$. Suppose that $\rho_{tt},\rho_{ttt}\in L^2(0,T;L^2(\Omega))$. Then, there exists a positive constant C independent of h and τ such that, for τ sufficiently small,*

$$\left\|\frac{\rho^n-\rho^{n-1}}{\tau}-\frac{\rho_h^n-\rho_h^{n-1}}{\tau}\right\| \leq C\big(h+\tau\big). \tag{4.34}$$

Proof. Taking the difference in time of (4.28a) and (4.28b) we find that

$$\left(\delta\bar{\vartheta}^i,\mathbf{v}\right)-\left(\delta\bar{\theta}^i,\nabla\cdot\mathbf{v}\right)=0,\quad \forall\mathbf{v}\in\mathcal{M}_h, \tag{4.35a}$$

$$\phi\left(\delta^2\theta^i,q\right)+\left(\nabla\cdot\delta\bar{\vartheta}^i,q\right)=\phi\left(\delta(\delta\rho^i-\rho_t^{i-\frac{1}{2}}),q\right)-\phi\left(\delta(\bar{\rho}_t^i-\rho_t^{i-\frac{1}{2}}),q\right),\quad \forall q\in\mathcal{R}_h. \tag{4.35b}$$

From the sum of Eq. (4.35b) with $q=\delta\bar{\theta}^i$ and Eq. (4.35a) with $\mathbf{v}=\delta\bar{\vartheta}^i$, applying the Young inequality, we obtain

$$\frac{\left\|\delta\theta^i\right\|^2-\left\|\delta\theta^{i-1}\right\|^2}{\tau}-\frac{2}{1+\tau}\left\|\delta\theta^i\right\|^2 \leq C\left(\left\|\delta(\delta\rho^i-\rho_t^{i-\frac{1}{2}})\right\|^2+\left\|\delta(\bar{\rho}_t^i-\rho_t^{i-\frac{1}{2}})\right\|^2\right).$$

Applying the discrete Gronwall's inequality implies

$$\begin{aligned}\|\delta\theta^n\|^2 &\leq \left\|\delta\theta^1\right\|^2+C\tau\sum_{i=2}^{n}\left\|\delta(\delta\rho^i-\rho_t^{i-\frac{1}{2}})\right\|^2+\left\|\delta(\bar{\rho}_t^i-\rho_t^{i-\frac{1}{2}})\right\|^2\\ &=\left\|\frac{\theta^1}{\tau}\right\|^2+C\tau\sum_{i=2}^{n}\left\|\delta(\delta\rho^i-\rho_t^{i-\frac{1}{2}})\right\|^2+\left\|\delta(\bar{\rho}_t^i-\rho_t^{i-\frac{1}{2}})\right\|^2.\end{aligned} \tag{4.36}$$

Thanks to (4.29),

$$\left\|\frac{\theta^1}{\tau}\right\|^2 \le C\tau^2 \int_0^{t_1} \|\rho_{tt}\|^2 + \|\rho_{ttt}\|^2\,dt. \tag{4.37}$$

Thanks to (4.19a) and (4.19b),

$$\sum_{i=2}^{n}\Big(\left\|\delta(\delta\rho^i - \rho_t^{i-\frac{1}{2}})\right\|^2 + \left\|\delta(\bar{\rho}_t^i - \rho_t^{i-\frac{1}{2}})\right\|^2\Big) \le C\tau \sum_{i=2}^{n}\int_{t_{i-2}}^{t_i} \|\rho_{tt}\|^2 + \|\rho_{ttt}\|^2\,dt. \tag{4.38}$$

Combining (4.36) with (4.37) and (4.38) gives

$$\begin{aligned}\|\delta\theta^n\|^2 &\le C\tau^2\int_0^{t_1}\|\rho_{tt}\|^2 + \|\rho_{ttt}\|^2\,dt + C\tau^2\sum_{i=2}^{n}\int_{t_{i-2}}^{t_i}\|\rho_{tt}\|^2 + \|\rho_{ttt}\|^2\,dt\\ &\le C\tau^2\sum_{i=2}^{n}\int_{t_{i-2}}^{t_i}\|\rho_{tt}\|^2 + \|\rho_{ttt}\|^2\,dt\end{aligned}$$

for all $n = 2, \ldots, N+1$.

Then

$$\|\delta\theta^n\| \le C\tau\Big(\sum_{i=1}^{n}\int_{t_{i-2}}^{t_i}\|\rho_{tt}\|^2 + \|\rho_{ttt}\|^2\,dt\Big)^{\frac{1}{2}}. \tag{4.39}$$

By the triangle inequality, (4.39) and (4.24),

$$\|\delta(\rho^n - \rho_h^n)\| \le \|\delta\theta^n\| + \|\delta\zeta^n\| \le C\tau\Big(\sum_{i=2}^{n}\int_{t_{i-2}}^{t_i}\|\rho_{tt}\|^2 + \|\rho_{ttt}\|^2\,dt\Big)^{\frac{1}{2}} + Ch\left\|\rho^i\right\|_1.$$

This proves (4.34). □

5 Numerical Results

In this section we carry out numerical experiments using mixed finite element based on the Crank–Nicolson scheme to solve problem (4.1a)–(4.1b) in two dimensional regions. For simplicity, the region of examples are unit square $\Omega = [0,1]^2$. We used FEniCS [22] to perform our numerical simulations. We divided the unit square into an $\mathcal{N}\times\mathcal{N}$ mesh of squares, each then subdivided into two right triangles using the UnitSquareMesh class in FEniCS. The triangularization in region Ω is uniform subdivision in each dimension. Our problem is solved at each time level starting at $t = 0$ until the final time $T = 1$. At time $T = 1$, we measured the L^2-errors of the density and the momentum. We obtain the convergence rates $r = \frac{\ln(e_i/e_{i-1})}{\ln(h_i/h_{i-1})}$ of finite approximation at eight levels with the discretization parameters $h \in \{1/2, 1/4, 1/8, 1/16, 1/32, 1/64, 1/128, 1/256\}$(the mesh size is actually $h\sqrt{2}$) respectively.

To test the convergence rates of the proposed algorithm, we choose the true solution of the problem (3.1a)–(3.1b) by

$$\begin{aligned}&\rho(\mathbf{x},t) = e^t\,x_1^2(1-x_1)x_2(1-x_2) \quad \text{and}\\ &\mathbf{m}(\mathbf{x},t) = \begin{bmatrix}-e^t\,x_1(2-3x_1)x_2(1-x_2)\\ -e^t\,x_1^2(1-x_1)(1-2x_2)\end{bmatrix} \quad \forall(\mathbf{x},t)\in[0,1]^2\times(0,1].\end{aligned}$$

For simplicity, we take $\phi = 1$ on Ω. The forcing term f is determined from equation $\rho_t + \nabla \cdot \mathbf{m} = f$. Explicitly,

$$f(\mathbf{x}, t) = e^t\, x_1^2(1 - x_1)x_2(1 - x_2) - 2e^t \left[(1 - 3x_1)x_2(1 - x_2) - x_1^2(1 - x_1)\right].$$

The initial condition and boundary condition are determined according to the analytical solution as follows:

$$\rho_0(\mathbf{x}) = x_1^2(1 - x_1)x_2(1 - x_2), \qquad \text{and} \qquad \rho(\mathbf{x}, t)\Big|_{\partial\Omega} = 0.$$

The numerical results are listed in Table 1 below.

$\mathcal{N}$	$\|\rho - \rho_h\|$	Rates	$\|\mathbf{m} - \mathbf{m}_h\|$	Rates
2	1.2462e-2	–	6.2211e-2	–
4	3.5764e-3	1.80	3.1202e-2	0.99
8	8.8489e-4	2.01	1.5252e-2	1.03
16	2.1890e-4	2.02	7.4188e-3	1.03
32	5.4645e-5	2.00	3.6974e-3	1.00
64	1.3666e-5	2.00	1.8452e-3	1.00
128	3.4149e-6	2.00	9.2200e-4	1.00
256	8.5373e-7	2.00	4.6035e-4	1.00

Table 1: *Results of the Crank–Nicolson scheme for the density and momentum with* $\tau = h/20$.

For the given problem, nearly second order and first order convergence are observed respectively in L^2 for the density and momentum. Slightly better than second order convergence is observed for the density, but as the mesh is refined, the error ratio approaches one in accordance with the theory.

In the second example we consider the non-homogeneous Dirichlet boundary condition. We test the stability of the method with different time steps. We take the true solution of the problem (3.1a)–(3.1b) to be

$$\rho(\mathbf{x}, t) = e^{-t}\, \sin \pi x_1\, \sin x_2, \quad \mathbf{m}(\mathbf{x}, t) = \begin{bmatrix} -\pi e^{-t}\, \cos \pi x_1\, \sin x_2 \\ -e^{-t}\, \sin \pi x_1\, \cos x_2 \end{bmatrix} \quad (\mathbf{x}, t) \in [0, 1]^2 \times (0, 1].$$

The forcing term f, initial condition and boundary condition accordingly are

$$f(\mathbf{x}, t) = \pi^2 e^{-t}\, \sin \pi x_1\, \sin x_2, \quad (\mathbf{x}, t) \in [0, 1]^2 \times [0, 1],$$

and

$$\rho_0(\mathbf{x}) = \sin \pi x_1\, \sin x_2, \quad \rho(\mathbf{x}, t)|_{\partial\Omega} = \begin{cases} 0 & \text{if } (x_1, x_2) \in \{0, 1\} \times (0, 1], \\ e^{-t}\, \sin 1\, \sin \pi x_1 & \text{if } (x_1, x_2) \in (0, 1) \times \{1\}. \end{cases}$$

Table 2 presents the results for $\tau = h$.

Tables 1-2 represent the numerical solution errors and convergence rates in the L^2-norm. In both cases, errors are calculated at time $T = 1$ and clearly demonstrate the second order of convergence for the density variable and first order of convergence for the momentum variable in the L^2-norm.

$\mathcal{N}$	$\|\rho-\rho_h\|$	Rates	$\|\mathbf{m}-\mathbf{m}_h\|$	Rates
2	1.6323e-2	–	4.1600e-01	–
4	4.3628e-3	1.90	2.1611e-01	0.94
8	1.0634e-3	2.04	1.1040e-01	0.97
16	2.6451e-4	2.01	5.5712e-02	0.99
32	6.5917e-5	2.00	2.7833e-02	1.00
64	1.6453e-5	2.00	1.3856e-02	1.00
128	4.1139e-6	2.00	6.9205e-03	1.00
256	1.0288e-6	2.00	3.4545e-03	1.00

Table 2: *Results of the Crank–Nicolson scheme for the density and momentum with $\tau = h$.*

6 Conclusion

In this paper, we have established a new fully discrete mixed finite element method based on the Crank–Nicolson scheme for Darcy flows. The spatial discretization is mixed and based on the lowest-order Raviart–Thomas finite elements, whereas the time discretization is based on the Crank–Nicolson scheme. We have proven the convergence of the scheme by estimating the error in term of discretization parameters. It has been shown that our method has the optimal convergence rate for the density and momentum and this scheme has stability with the different time steps. The numerical experiments agree with the estimates derived theoretically. Obviously, this method can be expanded to the case of many dimensions easily. There are some open questions including the possible extension of the method to non-Darcy fluid flows.

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