Sums of Two Generalized Tetrahedral Numbers

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Abstract - Expressing whole numbers as sums of figurate numbers, including tetrahedral numbers, is a longstanding problem in number theory. Pollock's tetrahedral number conjecture states that every positive integer can be expressed as the sum of at most five tetrahedral numbers. Here we explore a generalization of this conjecture to negative indices. We provide a method for computing sums of two generalized tetrahedral numbers up to a given bound, and explore which families of perfect powers can be expressed as sums of two generalized tetrahedral numbers.

Keywords : tetrahedral numbers; generalized tetrahedral numbers; figurate numbers; sums of figurate numbers; Waring's problem

Mathematics Subject Classification (2020) **:** 11P05

1 Introduction

A historically important problem in number theory, known as Waring's problem, is to represent whole numbers as sums of perfect powers, and in particular to find the minimum amount of such powers required to sum to any given whole number. For instance, it is known that every positive integer is the sum of at most 9 positive cubes ([4]). However, if we allow some of the cubes to be negative, this bound is considerably decreased, and it is conjectured that every integer is the sum of at most 4 integer cubes ([5]).

Similar problems to Waring's problem have been posed for other types of figurate numbers, including tetrahedral numbers. In 1850, Frederick Pollock conjectured that every positive integer can be expressed as the sum of at most five tetrahedral numbers ([6]), and the problem is still open. Recently, progress has been made on a generalization of this problem which allows negative indices for the tetrahedral numbers. If we let Te_n represent the nth tetrahedral number, $Te_n = \frac{n(n+1)(n+2)}{6}$, then for integer n, we call Te_n a "generalized tetrahedral number", or GTN. Note that $Te_0 = Te_{-1} = Te_{-2} = 0$, and that for all $n \in \mathbb{Z}$, $Te_{-n} = -Te_{n-2}$. So we see that the nonzero GTNs with negative indices are the opposites of the positive GTNs. Thus an equivalent generalization of the problem would be one allowing signed sums of tetrahedral numbers. Here we will instead use the negative index representation.

We say that an integer n is a sum of k GTNs if there exist integers $t_1, t_2 \cdots t_k$ such that $\sum_{i=1}^{k} Te_{t_i} = n$. Note that if n is a sum of k GTNs with indices $t_1, \cdots t_k$, then $-n$ is also a sum of k GTNs with indices $s_i = -t_i - 2$. In [1] and [2] it was found that $2 < k \leq 4$ for all $n \in \mathbb{N}$. In this paper, we will prove two central theorems about sums of two GTNs.

The first allows us to readily compute sums of two GTNs below a given bound, while the second involves representing perfect powers as sums of two GTNs.

To motivate our first result, let us imagine that we are trying to determine if a given whole number n is a sum of two GTNs. Our first step may be to determine whether n is a sum of two positive tetrahedral numbers. To do this, we can sum all pairs of positive tetrahedral numbers less than or equal to n and check if any of the sums are equal to n. We do not need to include tetrahedral numbers greater than n in this computation, because adding positive tetrahedral numbers to them will only increase the sum further beyond n. In other words, we have an upper bound on which tetrahedral numbers to include in our sums. The following theorem provides a similar bound on the sizes of two generalized tetrahedral numbers required to sum to a given whole number n.

Theorem 1.1 *Let $n \in \mathbb{N}$ such that $n = Te_a + Te_{-b}$ with $a, b > 0$. Then a and $b-1$ are at most $\lfloor \frac{\sqrt{1+8n}-1}{2} \rfloor$.*

Proof.

Let $n = Te_a + Te_{-b}$ with $a, b > 0$. Note that $b \leq a+1$, or else n would be zero or negative. Hence for a given a, the smallest possible positive value of $Te_a + Te_{-b}$ is $\frac{a(a+1)}{2}$, which occurs when b has its maximum value $a+1$.

Let m be the largest positive integer such that $Te_m + Te_{-m-1} \leq n$. Then we have

$$\frac{m(m+1)}{2} \leq n < \frac{(m+1)(m+2)}{2}.$$

The left-hand side becomes

$$m^2 + m - 2n \leq 0$$

$$m \leq \frac{\sqrt{1+8n}-1}{2},$$

while the right-hand side becomes

$$m^2 + 3m + 2 - 2n > 0$$

$$m > \frac{\sqrt{1+8n}-3}{2}.$$

Let $s = \sqrt{1+8n}$. Then $m \in (\frac{s-1}{2} - 1, \frac{s-1}{2}]$. Since m is an integer, we get

$$m = \left\lfloor \frac{s-1}{2} \right\rfloor.$$

Hence $a \leq \lfloor \frac{s-1}{2} \rfloor$. Since $b \leq a+1$, we also have $b - 1 \leq a \leq \lfloor \frac{s-1}{2} \rfloor$.

□

Corollary 1.2 *Let*

$$A_m = \{k \in \mathbb{N} : \exists a, b \in \mathbb{N}, (k = Te_a + Te_{-b}) \wedge (b - 1 \leq a \leq m)\}.$$

Then $n \in \mathbb{N}$ *is not a sum of two GTNs if both* $n \notin A_m$ *and* $n < \frac{(m+1)(m+2)}{2}$.

The above theorem allows us to easily conduct computer searches for sums of two GTNs, because it gives a limit on how far we need to search. For instance, if $n \leq 100$ is a sum of two GTNs, we see that the largest possible index for the GTNs is $\left\lfloor \frac{\sqrt{801}-1}{2} \right\rfloor = 13$. By summing all pairs of GTNs with indices between -14 and 13 and choosing those sums which are in the interval $[1, 100]$, we get all the sums of two GTNs below 100, shown below. The numbers which are sums of two positive tetrahedral numbers ([3]) form a subsequence, and are shown in bold.

1	**2**	3	**4**	**5**	6	**8**	9	**10**	**11**	**14**	15
16	19	**20**	**21**	**24**	25	28	**30**	31	34	**35**	**36**
39	**40**	**45**	46	49	52	**55**	**56**	**57**	**60**	64	**66**
70	74	**76**	78	80	81	83	**84**	**85**	**88**	**91**	**94**

Of these 48 sums of two GTNs, 29 of them, or 60.4%, are also sums of two positive tetrahedral numbers. The following table shows the values of this proportion when the sums of two GTNs have indices in the interval $[-m-1, m]$ for various values of m. For instance, we have just seen that when $m = 13$, this value is 60.4%. The value seems to decrease as m increases.

m	Proportion
100	44.4%
200	41.6%
300	40.0%
400	39.3%
500	38.7%

One may wonder whether there is a result analogous to Theorem 1.1 which would provide a bound on the indices when n is a sum of three GTNs. While such a result would undoubtedly be useful in future analysis, it is possible that no such bound exists. In the problem of sums of three integer cubes, it is often possible to add two large positive cubes and one large negative cube, or two large negative cubes and one large positive cube (where the indices are on the order of 10^{20}), to achieve sums as small as 3 ([7]). A similar situation may occur with tetrahedral numbers, so we will not attempt to find a bound on sums of three GTNs here.

2 Modular Analysis

It was shown in [2] that no prime numbers congruent to 2 (mod 5) apart from 2 are sums of two GTNs. Here we will analyze whether this result has an effect on the distribution of sums of two GTNs within each equivalence class modulo 5. We first notice that the index of a tetrahedral number modulo 5 determines its modulo 5 value according to the following table:

a mod 5	Te_a mod 5
0	0
1	1
2	4
3	0
4	0

We now let n be a sum of two GTNs, $n = Te_a + Te_b$. We wish to determine the probability of n to be in each equivalence class modulo 5. To that end, we assume that each of the indices a and b are chosen independently and uniformly from the modulo 5 equivalence classes. Under this assumption, there are 25 equally likely choices for the pair (a, b). We can then compute the values of n modulo 5 in each of these 25 cases and calculate the proportions which fall into each equivalence class. The resulting distribution, which we will refer to as the "theoretical distribution", is shown below. We see that we should expect 4% of sums of two GTNs to be congruent to 2 (mod 5).

n mod 5	Proportion
0	44%
1	24%
2	4%
3	4%
4	24%

The following tables show the counts and proportions of sums of two GTNs between 0 and m which are in each equivalence class modulo 5, for various values of m. They also show the difference (in percentage points) between the theoretical distribution and the actual distributions up to m. We see that 2 (mod 5) is indeed slightly less common than 3 (mod 5), possibly due to the lack of primes in this equivalence class. Also of note is the fact that multiples of 5 are considerably less common than expected.

$m = 10,000$:

n mod 5	Count	Proportion	Difference
0	599	39.18%	−4.82%
1	394	25.77%	+1.77%
2	69	4.51%	+0.51%
3	92	6.02%	+2.02%
4	375	24.53%	+0.53%

$m = 100,000$:

n mod 5	Count	Proportion	Difference
0	3239	41.17%	−2.83%
1	1982	25.19%	+1.19%
2	336	4.27%	+0.27%
3	401	5.1%	+1.1%
4	1910	24.28%	+0.28%

The accompanying Figure 1 shows the relative proportion of sums of two GTNs in each equivalence class with respect to m. Together with the above data tables, this suggests that the proportion of sums of two GTNs in each equivalence class will eventually approach a limit as m increases. Assuming the limiting values are close to the values when $m = 100,000$, they are likely within 1.5 percentage points of the theoretical distribution, except in the 0 (mod 5) equivalence class.

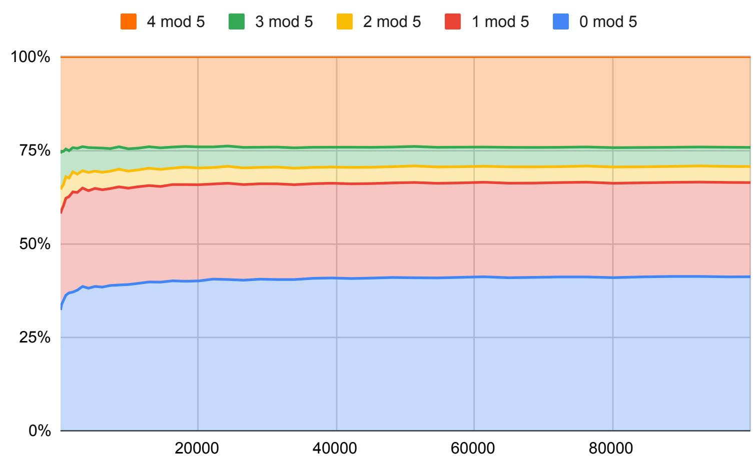

Figure 1: Stacked area chart showing the distribution of sums of two GTNs among the modulo 5 equivalence classes.

We will now conduct a similar analysis modulo 7. The theoretical distribution modulo 7 is as follows:

n mod 7	Proportion
0	26.53%
1	14.29%
2	6.12%
3	16.33%
4	16.33%
5	6.12%
6	14.29%

The following tables show the counts, proportions, and differences for sums of two GTNs between 0 and m which are in each equivalence class modulo 7, for various values of m, similar to the modulo 5 tables above.

$m = 10,000$:

n mod 7	Count	Proportion	Difference
0	354	23.15%	−3.38%
1	242	15.83%	+1.54%
2	114	7.46%	+1.34%
3	259	16.94%	+0.61%
4	248	16.22%	−0.11%
5	102	6.67%	+0.55%
6	210	13.73%	−0.56%

$m = 100,000$:

n mod 7	Count	Proportion	Difference
0	1920	24.40%	−2.13%
1	1195	15.19%	+0.9%
2	545	6.93%	+0.81%
3	1317	16.74%	+0.41%
4	1263	16.05%	−0.28%
5	504	6.41%	+0.29%
6	1124	14.29%	±0.0%

As in the modulo 5 case, we see that multiples of 7 are less common than expected. We conjecture that this will occur for any prime modulus p.

3 Perfect Powers

We now turn our attention to perfect powers which are sums of two GTNs. To begin, we prove that all squares are representable in this way. Since all even powers are also squares, this result shows that any even power is a sum of two GTNs.

Theorem 3.1 *For all $n \in \mathbb{N}$, n^2 is a sum of two GTNs.*

Proof. For any $n \in \mathbb{N}$, we have

$$\begin{aligned} Te_n + Te_{-n} &= \frac{1}{6}\big(n(n+1)(n+2) + (-n)(-n+1)(-n+2)\big) \\ &= \frac{1}{6}(n^3 + 3n^2 + 2n - n^3 + 3n^2 - 2n) = n^2 \end{aligned}$$

Hence n^2 is a sum of two GTNs. $\square$

We now move on to perfect cubes. We first conduct a computer search over all perfect cubes below 120^3 and find that the cubes of the following integers are sums of two GTNs:

$$1, 2, 4, 9, 11, 16, 25, 36, 49, 64, 81, 100, 109, 110$$

The majority of these cubes are also squares, *i.e.* they are sixth powers. Since we already know that all squares are sums of two GTNs, we wish instead to focus on non-square cubes. To this end, we introduce the following definition:

Definition 3.2 *We say that an integer p is a* pure nth power *iff $n \in \mathbb{N} \setminus \{1\}$ is minimal such that $p = a^n$ for some $a \in \mathbb{Z}$.*

We see that only 4 of the 14 cubes in the above list are pure, namely:

- $2^3 = Te_2 + Te_2$
- $11^3 = Te_{19} + Te_1$
- $109^3 = Te_{197} + Te_{19}$
- $110^3 = Te_{199} + Te_{-25}$

Based on the data above, it seems that it is relatively rare for a pure perfect cube to be a sum of two GTNs. In fact, we will now show that there are infinitely many pure perfect cubes which are not a sum of two GTNs. The proof references the following two lemmas, which will be proven shortly.

Lemma 3.3 *Let $n \in \mathbb{R}$ and $a \in \{1, 2, 3, 6\}$. Let $p \geq 3$. Then the equation*

$$3n^2 + 6n + 2 = \frac{6p}{a} - a^2p^4 + 3ap^2(n + 1) \tag{1}$$

does not hold.

Lemma 3.4 *Let $n \in \mathbb{N}$ and $a \in \{1, 2, 3, 6\}$. Let $p \in \mathbb{N}$ be congruent to 3 or 18 (mod 35). Then the equation*

$$3n^2 + 6n + 2 = \frac{6p^2}{a} - a^2p^2 + 3ap(n + 1) \tag{2}$$

does not hold.

We are now ready to prove the main theorem.

Theorem 3.5 *If p is prime and $p \equiv 3$ or $p \equiv 18$ (mod 35), then p^3 is not a sum of two generalized tetrahedral numbers.*

Proof. Suppose that p is prime and $p \equiv 3$ or $p \equiv 18 \pmod{35}$. By way of contradiction, suppose also that p^3 is a sum of two generalized tetrahedral numbers. Then either both GTNs have positive indices, or exactly one GTN has a negative index. We consider each of these cases in turn.

Suppose p^3 can be written as a sum of two GTNs with positive indices. Then there exist integers $n \geq k \geq 0$ such that

$$p^3 = Te_n + Te_{n-k}.$$

From here we get

$$6p^3 = (2n - k + 2)(k^2 - kn - k + n^2 + 2n).$$

Let $A = 2n - k + 2, B = k^2 - kn - k + n^2 + 2n$. Since p is prime, one of the following cases holds:

1. $p^3 \mid A$
2. $p^3 \mid B$
3. $p^2 \mid A$ and $p \mid B$
4. $p \mid A$ and $p^2 \mid B$

Case 1: $p^3 \mid A$ and $B \mid 6$. Note that $B = n(n-k) + k(k-1) + 2n \geq 2n$, so $n \leq 3$. We then get $A = 2n - k + 2 \leq 8$, so we must have $p = 2$. Since 2 is not congruent to 3 or 18 (mod 35), we may disregard it here.

Case 2: $p^3 \mid B$ and $A \mid 6$. Then $(n, k) \in \{(0,0), (1,1), (2,0), (3,2), (4,4)\}$. The only pair that gives a perfect-cube value for p^3 is (2,0), giving $p = 2$, which we may disregard.

Case 3: $p^2 \mid A$ and $p \mid B$. Then $A = 2n - k + 2 = ap^2$ for some $a \mid 6$. Substituting into the expression for B gives

$$3n^2 - 3ap^2(n+1) + 6n + a^2p^4 + 2 = \frac{6p}{a}$$

$$3n^2 + 6n + 2 = \frac{6p}{a} - a^2p^4 + 3ap^2(n+1)$$

By Lemma 3.3, $p < 3$, but then p cannot be congruent to 3 or 18 (mod 35).

Case 4: $p \mid A$ and $p^2 \mid B$. Then $A = 2n - k + 2 = ap$ for some $a \mid 6$. Substituting into the expression for B gives

$$3n^2 - 3anp + 6n + a^2p^2 - 3ap + 2 = \frac{6p^2}{a}$$

$$3n^2 + 6n + 2 = \frac{6p^2}{a} - a^2p^2 + 3ap(n+1)$$

By Lemma 3.4, this case is impossible. Hence p^3 is not a sum of two positive GTNs.

Now suppose that p^3 is a sum of two GTNs with exactly one of the indices negative. Then there exist integers $n \geq 0$ and $-1 \leq k \leq n$ such that

$$p^3 = Te_n + Te_{-(n-k)}.$$

From here we get

$$6p^3 = (k+2)(3n^2 - 3kn + k^2 + k).$$

Let $A = k+2, B = 3n^2 - 3kn + k^2 + k$. Since p is prime, one of the following cases holds:

1. $p^3 \mid A$
2. $p^3 \mid B$
3. $p^2 \mid A$ and $p \mid B$
4. $p \mid A$ and $p^2 \mid B$

Case 1: $p^3 \mid A$ and $B \mid 6$. Note that $B = 3n(n-k) + k(k+1) \geq k(k+1)$. So $k \leq 2$, and $A = k+2 \leq 4$. Since $p \geq 2$, this contradicts $p^3 \mid A$.

Case 2: $p^3 \mid B$ and $A \mid 6$. Here we follow an argument similar to the one used in [2]. Note that because $p \equiv 3 \pmod 5$, we have $p^3 \equiv 2 \pmod 5$. Now $A = k+2 \in \{1,2,3,6\}$. If $k=0$ then we get $p^3 = n^2$, impossible because p is prime. If $k=1$ then we get

$$\begin{aligned} 3n^2 - 3n + 2 = 2p^3 &\equiv_5 4 \\ 3n^2 - 3n + 3 = 3(n^2 - n + 1) &\equiv_5 0 \\ n^2 + 4n + 4 = (n+2)^2 &\equiv_5 3, \end{aligned}$$

which is impossible. If $k=4$ then we get

$$\begin{aligned} 3n^2 - 12n + 20 = p^3 &\equiv_5 2 \\ 3n^2 - 12n + 12 = 3(n^2 - 4n + 4) &\equiv_5 4 \\ n^2 - 4n + 4 = (n-2)^2 &\equiv_5 3, \end{aligned}$$

which is also impossible. If $k = -1$ then we get

$$p^3 = Te_n + Te_{-n-1} = \frac{1}{6}(n^3 + 3n^2 + 2n - n^3 + n) = \frac{n^2+n}{2}$$

$$8p^3 = (2p)^3 = 4n^2 + 4n = (2n+1)^2 - 1$$

$$(2n+1)^2 - (2p)^3 = 1.$$

By Catalan's conjecture (proven in [8]), the only solution to the above occurs when $p = 1$, which we are not considering here.

Case 3: $p^2 \mid A$ and $p \mid B$. Hence $A = k + 2 = ap^2$ for some $a \mid 6$. Substituting into the expression for B gives

$$B = 3n^2 - 3n(ap^2 - 2) + (ap^2 - 2)^2 + ap^2 - 2 = \frac{6p}{a}$$

$$3n^2 + 6n + a^2p^4 + 2 - 3ap^2(n+1) = \frac{6p}{a}$$

$$3n^2 + 6n + 2 = p(\frac{6}{a} - a^2p^3 + 3ap(n+1))$$

By Lemma 3.3, $p < 3$, but then p cannot be congruent to 3 or 18 (mod 35).

Case 4: $p \mid A$ and $p^2 \mid B$. Hence $k = ap - 2$ for some $a \mid 6$. Substituting into the expression for B gives

$$3n^2 - 3anp + 6n + a^2p^2 - 3ap + 2 = \frac{6p^2}{a}$$

$$3n^2 + 6n + 2 = \frac{6p^2}{a} - a^2p^2 + 3ap(n+1)$$

By Lemma 3.4, this case is impossible. Hence p^3 is not a sum of two GTNs.

□

Proof. [Proof of Lemma 3.3]

Suppose that equation (1) holds under these conditions. We first rearrange it as follows:

$$3n^2 + (6 - 3ap^2)n + 2 = \frac{6p}{a} - a^2p^4 + 3ap^2$$

The left-hand side is a quadratic in n; its minimum value is obtained when $n = \frac{1}{2}ap^2 - 1$. So we get

$$3\big(\frac{1}{2}ap^2 - 1\big)^2 + (6 - 3ap^2)\big(\frac{1}{2}ap^2 - 1\big) + 2 \leq \frac{6p}{a} - a^2p^4 + 3ap^2$$

$$-\frac{3}{4}a^2p^4 + 3ap^2 - 1 \leq \frac{6p}{a} - a^2p^4 + 3ap^2$$

$$\frac{6p}{a} + 1 \geq \frac{1}{4}a^2p^4$$

Since $p \geq 3$ and $a \geq 1$, we get

$$6p + 1 \geq \frac{6p}{a} + 1 \geq \frac{1}{4}a^2p^4 \geq \frac{1}{4}p^4 \geq \frac{27}{4}p$$

So $1 \geq \frac{3}{4}p$ or $p \leq \frac{4}{3}$. By contradiction, equation (1) does not hold.

□

Proof. [Proof of Lemma 3.4]

Note that $p \equiv 3 \pmod 5$. Reducing equation (2) modulo 5 gives

$$3n^2 + n + 2 \equiv_5 3(\frac{3}{a} - 3a^2 + 3an + 3a) \equiv_5 \frac{-1}{a} + a^2 - an - a$$

If $a \equiv_5 1$, then this becomes $3n^2+n+2 \equiv_5 -1-n \Rightarrow 3n^2-3n+3 \equiv_5 n^2-n+1 \equiv_5 0$, which is impossible. If $a \equiv_5 2$, then we get $3n^2+n+2 \equiv_5 4-2n \Rightarrow 3n^2+3n+3 \equiv_5 n^2+n+1 \equiv_5 0$, which is also impossible. So we see that $a \notin \{1, 2, 6\}$.

If $a = 3$, then equation (2) becomes

$$3n^2 + 6n + 2 = p(-7p + 9n + 9)$$

Reducing modulo 7 gives

$$3n^2 + 6n + 2 \equiv_7 p(2n + 2)$$
$$5n^2 + 3n + 1 \equiv_7 p(n + 1)$$

Note that either $p \equiv 3 \pmod 7$ or $p \equiv 4 \pmod 7$. If $p \equiv 3 \pmod 7$, then we get

$$5n^2 + 3n + 1 \equiv_7 3n + 3$$
$$5n^2 + 5 \equiv_7 0$$
$$n^2 \equiv_7 6,$$

which is impossible. If $p \equiv 4 \pmod 7$, then we get

$$5n^2 + 3n + 1 \equiv_7 4n + 4$$
$$5n^2 + 6n \equiv_7 3$$
$$n^2 + 4n + 4 = (n + 2)^2 \equiv_7 6,$$

which, again, is impossible. So we see that $a \notin \{1, 2, 3, 6\}$, and equation (2) does not hold.

□

Theorem 3.5 provides two infinite families of pure cubes which are never sums of two GTNs. However, it does not cover all the pure cubes that are not sums of two GTNs. For instance, Theorem 3.5 alone does not prevent 5^3 from being a sum of two GTNs, but we have seen that it is not. There are likely many more families of pure cubes that are never sums of two GTNs, which would help to explain the observed scarcity of such cubes.

We conclude with a brief discussion of fifth powers. A computer search showed that there are no pure fifth powers less than or equal to 30^5 which are sums of two GTNs. However, as we have seen in the case of cubes, the pure perfect powers which are sums of two GTNs are relatively sparse. As such, we are not prepared to conjecture that no pure fifth powers are sums of two GTNs. We have also not managed to prove a result analogous to Theorem 3.5 for fifth powers; it is still an open question whether there are infinitely many fifth powers which are not sums of two GTNs.

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