Blowup of Solutions to a Damped Euler Equation with Homogeneous Three-Point Boundary Condition

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Abstract - It is known that solutions to the inviscid Proudman-Johnson equation subject to a homogeneous three-point boundary condition can develop singularities in finite time. Given the regularizing effect that damping can have, in this paper we consider the possibility of finite-time singularity formation in solutions of the generalized inviscid Proudman-Johnson equation with damping subject to the same homogeneous three-point boundary condition. In particular, we will derive criteria the initial condition must satisfy in order for solutions to blowup in finite time under the effect of a smooth time-dependent damping term.

Keywords : generalized Proudman-Johnson equation; blowup; three-point boundary condition; damping

Mathematics Subject Classification (2020) **:** 35B44; 35B65; 35Q35

1 Introduction

We study the initial value problem (IVP)

$$\begin{cases} u_{xxt} + uu_{xxx} + \beta u_x u_{xx} + \alpha(t) u_{xx} = 0, & x \in [0,1],\ t > 0, \\ u(x,0) = u_0(x), & x \in [0,1], \end{cases} \tag{1}$$

where $\beta = \frac{n-3}{n-1}$, $n \in \mathbb{Z}^+$, $n \geq 2$, and $u(x,t)$ satisfies the homogeneous three-point boundary condition

$$u(1,t) = u_x(0,t) = u_x(1,t) = 0 \tag{2}$$

Equation

$$u_{xxt} + uu_{xxx} + \beta u_x u_{xx} + \alpha(t) u_{xx} = 0 \tag{3}$$

can be obtained by imposing on the incompressible $n-$dimensional Euler equations with damping

$$\boldsymbol{u}_t + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + \alpha(t)\Delta \boldsymbol{u} = -\nabla p, \qquad \nabla \cdot \boldsymbol{u} = 0$$

velocities of the form

$$\boldsymbol{u}(x, \boldsymbol{x}', t) = (u(x,t), -\frac{\boldsymbol{x}'}{n-1} u_x(x,t))$$

for $\boldsymbol{x}' = \{x_2, ..., x_n\}$, or by using the cylindrical coordinate representation

$$u^r = -\frac{r}{n-1}\,u(x,t), \quad u^\theta = 0, \qquad u^x = u(x,t)$$

where $r = |\boldsymbol{x}'|$ ([4], [22], [20], [14], [6], [16]).

We will refer to (1)-(2) as the initial boundary value problem (IBVP) for the generalized inviscid Proudman-Johnson equation with damping.

For $\beta \in \mathbb{R}$, we remark that the undamped version of (3), i.e.

$$u_{xxt} + uu_{xxx} + \beta u_x u_{xx} = 0 \tag{4}$$

arises in several additional physical and geometrical contexts. For instance, when $\beta = 3$, (4) reduces to the Burgers' equation of gas dynamics. For $\beta = 2$, it becomes the Hunter-Saxton equation (HS) describing the orientation of waves in massive nematic liquid crystals ([9], [3], [5], [24]). From a more geometric point of view, periodic solutions to the HS equation also describe geodesics on the group $\mathcal{D}(\mathbb{S})\backslash Rot(\mathbb{S})$ of orientation preserving diffeomorphisms on the unit circle modulo rigid rotations with respect to the right-invariant metric ([10], [3], [21], [11])

$$\langle f, g \rangle = \int_{\mathbb{S}} f_x g_x dx$$

Furthermore, Lenells and Misiolek [12] showed that, for any β, (4) arises as the geodesic equation of the affine connection $\nabla^{(\alpha)}$ on the group $\mathcal{D}(\mathbb{S})\backslash Rot(\mathbb{S})$. See also [2] for yet another derivation of (4) as a geodesic equation.

From a more heuristic point of view, (3) may serve as a tool to better understand the role that convection and stretching play in the regularity of solutions to one-dimensional fluid evolution equations; it has been argued that the convection term can sometimes cancel some of the nonlinear effects and contribute positively to regularity of solutions ([13], [8], [15]). More particularly, setting $\omega = u_{xx}$ in (3) yields

$$\omega_t + \underbrace{u\omega_x}_{\text{convection}} + \beta \underbrace{\omega u_x}_{\text{stretching}} + \alpha(t)\omega = 0 \tag{5}$$

In the undamped case ($\alpha(t) \equiv 0$) and $\beta = -1$, (5) becomes a one-dimensional analogue of the three-dimensional vorticity equation of incompressible inviscid fluids

$$\boldsymbol{\omega}_t + \boldsymbol{u} \cdot \nabla \boldsymbol{\omega} = \boldsymbol{\omega} \cdot \nabla \boldsymbol{u}, \qquad \boldsymbol{\omega} = \nabla \times \boldsymbol{u}$$

The nonlinear terms in equation (5) represent the competition between nonlinear convection and stretching ([7], [23]). More particularly, the parameter $\beta \in \mathbb{R}$ is related to the ratio of stretching to convection.

2 Previous Regularity Results

The global regularity of solutions of the damped equation (3) for a particular value of β is discussed in [19] for solutions satisfying periodic boundary conditions. In the undamped case, finite-time singularity formation has been extensively studied for periodic solutions as well as Dirichlet boundary conditions (see e.g. [17], [18] and references therein). However, we are unaware of any results concerning singularity formation in solutions of the damped equation (3) with the homogeneous three-point boundary condition (2). In the undamped case with homogeneous three-point boundary condition (2), the only result we are aware of is Theorem (2.1) below established by Yuen ([1]) where conditions on the initial data leading to finite-time blowup are derived for $\beta = -1$. Therefore, it is of interest to study whether finite-time blowup of solutions is still a possibility after incorporating damping into the system as well as determining if varying the value of the parameter β has any effect on the regularity of solutions which, in turn, could lead to a better understanding of the competing effects between nonlinear convection and stretching as discussed at the end of the previous section.

Theorem 2.1 *(Yuen [1]) Consider the C^3 solutions to the IBVP (1)-(2) for the inviscid undamped Proudman Johnson equation ($n = 2$, $\alpha(t) \equiv 0$). If the initial velocity satisfies*

$$U_0 = \int_0^1 u_0'(x)\, dx = -u_0(0) > 0$$

then the solutions blowup on or before the finite time $1/(2U_0)$.

The outline of the paper is a follows. In Section 3, we prove finite-time blowup of solutions to (1)-(2) with arbitrary smooth and bounded time-dependent damping term, while the formation of singularities in finite time with smooth and unbounded damping term is discussed in Section 4.

3 Finite-time Blowup with Smooth, bounded Damping

In this section, we will derive conditions on the initial data $u_0(x)$ which show that the presence of a smooth (bounded or unbounded) time-dependent damping term $\alpha(t)$ is insufficient to arrest finite-time blowup for any value of the parameter $\beta \in \mathbb{R}$.

Theorem 3.1 *Consider the IBVP (1)-(2) for the generalized inviscid damped Proudman Johnson equation with $\beta = \frac{n-3}{n-1}$, $n \in \mathbb{Z}^+$, $n \geq 2$, initial data $u_0(x) \in C^\infty([0,1])$, and smooth bounded damping term $\alpha(t) : [0, \infty) \to \mathbb{R}^+$.*

Set

$$H_0 = -\int_0^1 u_0'(x)\, dx = u_0(0)$$

and

$$M := \sup_{t \in [0,\infty)} \alpha(t) \tag{6}$$

for some $M \in \mathbb{R}^+$. If

$$H_0 < M \left(\frac{1-n}{n} \right), \tag{7}$$

then

$$\lim_{t \nearrow t^*} u(0,t) = -\infty \tag{8}$$

for positive t^* *given by*

$$t^* = -\frac{1}{M} \ln\left(1 - \frac{M(1-n)}{nH_0}\right)$$

Proof.

Multiplying (3) by x, integrating over $x \in [0,1]$ and using (2) gives

$$H'(t) + \alpha(t)H(t) + \frac{n}{n-1}\|u_x(\cdot,t)\|_2^2 = 0 \tag{9}$$

for

$$H(t) = -\int_0^1 u_x(x,t)\,dx = u(0,t) \tag{10}$$

and where $\|u_x(\cdot,t)\|_2$ represents the $L^2([0,1])$ norm of u_x, i.e.

$$\|u_x(\cdot,t)\|^2_{L^2([0,1])} = \int_0^1 (u_x(x,t))^2\,dx$$

Now, the Cauchy-Schwarz inequality implies that

$$H^2 \le \|u_x(\cdot,t)\|_2^2$$

and so equation (9) yields the inequality

$$H'(t) + \alpha(t)H(t) + \frac{n}{n-1}H^2 \le 0 \tag{11}$$

Setting $f = H^{-1}$ in (11) and using an integrating factor argument we obtain

$$\frac{d}{dt}\left(f(t)e^{-\int_0^t \alpha(s)\,ds}\right) \ge \frac{n}{n-1}e^{-\int_0^t \alpha(s)\,ds}$$

which gives, after integrating,

$$\frac{1}{H(t)} \ge \frac{n(1+H_0 g(t))}{(n-1)H_0 g'(t)} \tag{12}$$

for $H_0 = H(0)$ and

$$g(t) = \frac{n}{n-1}\int_0^t e^{-\int_0^s \alpha(z)\,dz}\,ds \tag{13}$$

Next, from (6),

$$\alpha(t) \le M$$

for all $t \in [0, \infty)$ and some $M \in \mathbb{R}^+$. Consequently,

$$g'(t) \geq \frac{n}{n-1} e^{-Mt}$$

and, after integrating,

$$g(t) \geq \frac{n}{n-1} \left(\frac{1 - e^{-Mt}}{M} \right)$$

Using these inequalities on (12) we obtain

$$\frac{1}{H(t)} \geq \Lambda(t) \tag{14}$$

for

$$\Lambda(t) = \frac{nN(t)}{(n-1)^2 M H_0 g'(t)} \tag{15}$$

and

$$N(t) = M(n-1) + nH_0(1 - e^{-Mt})$$

Now, (7)ii) implies that

$$\lim_{t \to \infty} N(t) = M(n-1) + nH_0 < 0$$

This, along with

$$N(0) = M(n-1) > 0, \quad N'(t) = nMH_0 e^{-Mt} < 0 \quad \text{and} \quad N(t) \in C([0, \infty))$$

implies the existence of a finite time $t^* > 0$ such that

$$\lim_{t \nearrow t^*} N(t) = 0 \tag{16}$$

In turn, the above argument implies that $\Lambda(t) < 0$ when $t \in [0, t^*)$ and

$$\lim_{t \nearrow t^*} \Lambda(t) = 0$$

Finally, (11) gives

$$H'(t) + \alpha(t) H(t) \leq 0$$

which yields

$$H(t) \leq H_0 \, e^{-\int_0^t \alpha(s) ds} \tag{17}$$

Since $H_0 < 0$, (17) implies that $H(t) < 0$ for as long as it exists. Thus

$$0 \geq \frac{1}{H(t)} \geq \Lambda(t) \tag{18}$$

and the Theorem follows in the limit as t approaches t^* in (18).

□

4 Finite-time Blowup with Smooth, unbounded Damping

In Theorem 3.1 of the previous section, finite-time blowup was established for smooth and bounded time-dependent damping. In Theorem 4.1 below we show that finite-time blowup is still possible with smooth but unbounded time-dependent damping.

Theorem 4.1 *Consider the IBVP (1)-(2) for the generalized inviscid damped Proudman Johnson equation with* $\beta = \frac{n-3}{n-1}$, $n \in \mathbb{Z}^+$, $n \geq 2$, *and initial data* $u_0(x) \in C^\infty([0,1])$. *Let* $E_1(x)$ *be the exponential integral*

$$E_1(x) = \int_x^\infty \frac{e^{-t}}{t} dt, \qquad x > 0 \tag{19}$$

If

$$u_0(0) < -\frac{c(n-1)}{ne^{1/c}E_1(1/c)} \tag{20}$$

for $c = \alpha'(0) \in \mathbb{R}^+$, *then there exists smooth, unbounded damping* $\alpha(t) : [0, \infty) \to \mathbb{R}^+$ *and a finite time* $t^* > 0$ *such that*

$$\lim_{t \nearrow t^*} u(0,t) = -\infty \tag{21}$$

Proof.

Suppose

$$H_0 = -\int_0^1 u_0'(x)\, dx = u_0(0) < 0$$

From (12) and (17) we have that

$$0 > \frac{1}{H(t)} \geq \frac{n(1 + H_0 g(t))}{(n-1)H_0 g'(t)} \tag{22}$$

for $g(t)$ as in (13) and $n \in \mathbb{Z}^+$, $n \geq 2$. Set

$$\alpha(t) = e^{ct}$$

for some $c \in \mathbb{R}^+$. Then, after some straightforward computations, we can write (22) as

$$0 > \frac{1}{H(t)} \geq \frac{1 + H_0\phi(n)\eta(t)}{e^{\frac{1}{c}} H_0 e^{-\frac{e^{ct}}{c}}} \tag{23}$$

for

$$\phi(n) = \frac{ne^{\frac{1}{c}}}{c(n-1)}$$

and

$$\eta(t) = \int_{\frac{1}{c}}^{\frac{e^{ct}}{c}} \frac{e^{-u}}{u}\, du$$

Note that $\eta(0) = 0$, while $\eta'(t) > 0$ and $\eta''(t) < 0$ for all $t \in [0, \infty)$. Moreover, for the exponential integral $E_1(x)$ as defined in (19), we have that

$$\lim_{t\to\infty} \eta(t) = E_1(1/c)$$

The above remarks, along with the fact that the denominator of the right hand-side of (23) is negative for all $t \in [0, \infty)$, imply that if $H_0 = u_0(0)$ satisfies (20), then there exists a finite time $t^* > 0$ such that (21) follows.

□

Remark 4.2 Below are some sample upper bounds (20) (computed using Mathematica) for the initial data H_0 in Theorem 4.1 for certain values of n and c.

- If $n = 2$ and $c = 1$, then

$$H_0 < -\frac{1}{2eE_1(1)} \approx -0.84$$

- If $n = 2$ and $c = 0.01$, then

$$H_0 < -\frac{0.01}{2e^{100}E_1(100)} \approx -0.51$$

5 Conclusions and Open Questions

In this paper, we derived conditions on the initial data which guarantee the formation of singularities in finite time in solutions of the generalized inviscid Proudman-Johnson when arbitrary smooth and bounded damping is incorporated into the system. Moreover, we have shown that there exist smooth (but unbounded) damping terms for which finite-time blowup is also possible. It is unknown if solutions will persist for all time (instead of blowing up in finite time) if the initial data satisfies the opposite of inequality (7)ii) in Theorem 3.1, namely, if

$$H_0 \geq M\left(\frac{1-n}{n}\right)$$

It is also of interest to investigate how solutions to other fluid-related models such as the Boussinesq equations or the Magneto-Hydrodynamic (MHD) equations behave under a similar homogeneous three-point boundary type condition.

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