An IFS for the Stretched Level-n Sierpinski Gasket

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Abstract - Broadly, fractals are sets that exhibit a repeating pattern at multiple scales. One important fractal set is the Sierpinski Gasket (SG) which is made up of nested equilateral triangles. A variation of the classic Sierpinski gasket is to create n-levels with the equilateral triangles. Another variation is to stretch the points of intersection for the triangles in SG into line segments of length $0 < \alpha < 1/3$. When one combines these variations, one arrives at the stretched level-n Sierpinski gaskets (SSG^n) which are the focus of this work. We give an introduction to iterated function systems (IFS) and determine an IFS which generates SSG^3. We then describe how one can acquire the IFS for SSG^n in general, and conclude with a theorem which determines the Hausdorff dimension for SSG^n.

Keywords : fractals; Hausdorff dimension; Sierpinski gasket; iterated function system

Mathematics Subject Classification (2020) **:** 28A80; 11K55; 11S82

1 Introduction

Fractals are sets that exhibit a repeating pattern at multiple scales. A fractal can be perfectly self-similar (in the sense defined in Section 1) or can simply be a set which repeats a pattern at various scales, like the Weierstrass curve. The best way to understand fractal sets is to analyze various examples. In this work, we will focus on fractal sets which are self-similar or self-affine. Figure 1 shows three fractal sets, each of which exhibits self-similarity at finer and finer scales. Should we have a way to zoom further and further into these sets, we would find the same overall structure repeating infinitely.

Notice the remarkable relationship between the Barnsley fern and the fractal tree in Figure 1, and the patterns we observe in nature. Nature produces many fractal patterns. The reader is encouraged to find examples of fractal sets from within their own garden. Understanding fractal sets can help develop tools to study the complex patterns created by nature. This is one of the motivations behind the study of fractal geometry; see [3], [4], [5].

Fractals like the Sierpinski gasket, SG, shown in Figure 1 have a robust structure and serve as useful toy models to develop the theory of fractal geometry. This theory can then be extended to the fractals that arise in nature. In this work we explore fractal sets that arise from combining two variations on the classical Sierpinski gasket. The first variation is the addition of levels to form the level-n Sierpinski gasket, SG^n, seen in Figure 3.

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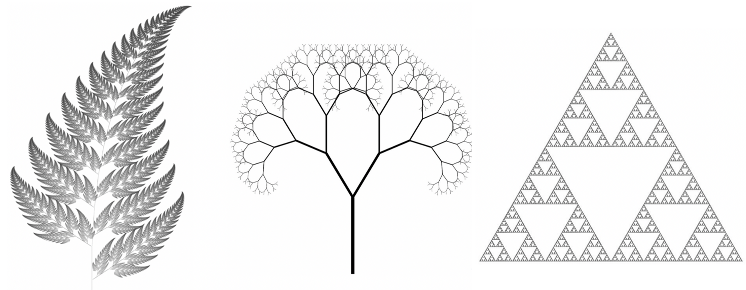

Figure 1: The Barnsley fern (left) [1], a fractal tree (center) [2] and the Sierpinski gasket (right).

The second variation is a *stretching* of the points of intersection for the triangles in SG. From a fractal geometric point of view, the Sierpinski gasket and the stretched Sierpinski gasket are distinct. At the core of that distinction is the fact that the Sierpinski gasket can be generated with contraction similarities (maps which shrink a space by the same ratio in all directions) and the stretched Sierpinski gasket is generated with contraction affine maps. Self-affine fractal sets are notoriously more difficult to work with, and the stretched Sierpinski gasket has been the subject of multiple papers in recent years; see for example [6], [7], [8].

By stretching the level-n Sierpinski gasket we get what is known as the stretched level-n Sierpinski gasket, SSG^n. The spaces SSG^n have not been thoroughly studied by fractal geometers. This work describes the way in which SSG^n is built via an iterated function system (IFS). In the process, we review important results about fractals arising from an IFS. We finish with a theorem describing the fractal (Hausdorff) dimension of SSG^n.

The remainder of this article is organized as follows:

- Section 2 introduces the concept of an iterated function system and the idea of fractal dimension. This section contains many important theorems from fractal geometry found in [9].

- Section 3 reviews the IFS that generates the stretched Sierpinski gasket.

- Section 4 builds upon the construction in section 3. This section explicitly constructs the IFS for the level-3 stretched Sierpinski gasket and describes how the construction extends for the level-n stretched Sierpinski gasket.

- In Section 5 we calculate the Hausdorff dimension of the stretched level-n Sierpinski gasket. For $n \in \mathbb{N}$ with $n \geq 2$ and $\alpha \in (0, \frac{1}{n+1})$ we have the following:

$$\dim_H SSG^n = \frac{\ln\left(\frac{n(n+1)}{2}\right)}{\ln(n) - \ln(1-(n-1)\alpha)}.$$

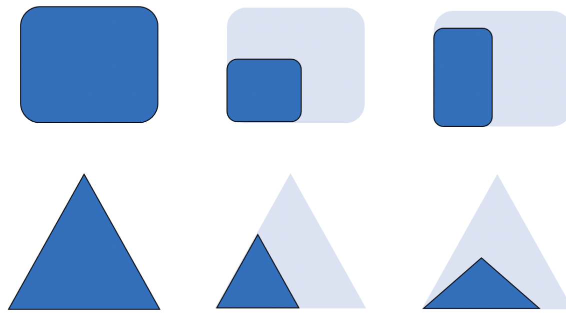

Figure 2: Above we have a square and triangle (left) and apply a similarity (center) and a generic contraction which is not a similarity (right). The "shadow" of the original square and triangle are included as a reference for both the similarity and generic contraction.

2 A brief introduction to iterated function systems (IFS)

In this section we review some fundamental definitions and results about fractals generated by iterated function systems.

Definition 2.1 *A function $f : D \subseteq \mathbb{R}^N \to \mathbb{R}^N$ is a* ***contraction*** *if there exists a real number $c \in [0, 1)$ such that*

$$|f(x) - f(y)| \leq c|x - y|$$

for all $x, y \in D$. A contraction $f : D \subseteq \mathbb{R}^N \to \mathbb{R}^N$ which satisfies $|f(x) - f(y)| = c|x - y|$ for all $x, y \in D$, is called a ***similarity****.*

Let $D \subseteq \mathbb{R}^N$ be a closed set. A finite collection of contractions $f_i : D \to D$ where $i = 1, 2, \ldots, k$ is called an ***iterated function system (IFS)****.*

Let's unpack the definitions above. Contraction maps and similarities are shrinking maps! The inequality in the definition of a contraction tells us that contractions shrink shapes and it is possible that the shrinking is greater in one direction than another. Similarities on the other hand shrink by the same amount in all directions; see Figure 2.

Now how exactly does an IFS generate a fractal set? This is a key theorem from fractal geometry which can be found in [9].

Theorem 2.2 *Let $g_1, g_2, \ldots, g_k$ be contractions on a closed set $D \subseteq \mathbb{R}^N$. There exists a unique non-empty compact set X that satisfies*

$$\bigcup_{i=1}^{k} g_i(X) = X.$$

The set X is called the ***attractor*** *of the IFS, $\{g_i : i = 1, 2, \ldots, k\}$.*

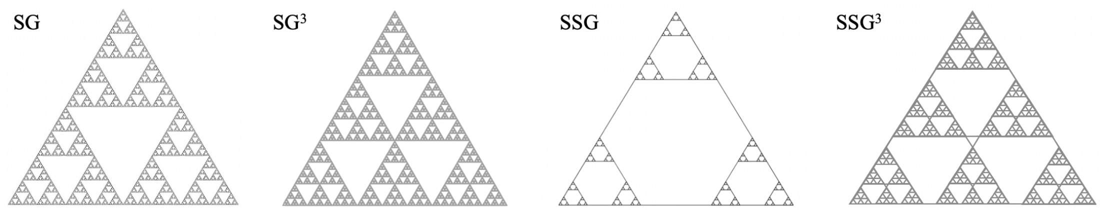

Figure 3: The Sierpinski gasket (SG) along with its level-3 (SG^3), stretched (SSG), and stretched level-3 (SSG^3) variations.

Moreover, define $F(A) = \bigcup_{i=1}^{k} g_i(A)$ *and write* $F^j(A) = F(F(F(\cdots F(A))))$ *for the* j*-th application of* F*. Then we have*

$$X = \bigcap_{j=1}^{\infty} F^j(E)$$

for any compact set E *such that* $g_i(E) \subseteq E$ *for each* $i = 1, 2, \ldots, k$*.*

The first part of the theorem above tells us that if we have an IFS, then there is exactly one non-empty compact set X which is left fixed when we apply and union the images of the contractions in the IFS. This is how we formalize the idea that X exhibits the same pattern at multiple scales. Given that we have

$$X = g_1(X) \cup g_2(X) \cup \cdots \cup g_k(X),$$

we see that X is made up of contracted (shrunken) copies of itself.

The second part of the theorem tells us how to visualize the construction of the set X. For example, in the case of the Sierpinski gasket we have

$$SG = F(E) \cap F^2(E) \cap F^3(E) \cap \cdots$$

where we choose the set E be the filled in equilateral triangle formed by the points

$$p_1 = (0, 0) \qquad p_2 = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) \qquad p_3 = (1, 0).$$

In this case, the set $F(E)$ is the first stage in the construction of SG, the set $F^2(E)$ is the second stage, and so on. We get SG by intersecting all of these stages in the construction process. For more on iterated function systems and fractal geometry in general see [9] and [10]. One quality that makes fractals arising from an IFS particularly special to work with is the understanding we have of the dimension of these fractals.

2.1 Hausdorff dimension

We may be accustom to thinking only of sets in dimensions 0, 1, 2, 3, etc., but there are interesting fractal sets living in the dimensions in between! Fractal sets often have non-integer dimensions.

There are various ways to extend our usual ideas of dimension to ones that may yield non-integer dimensions. Some notions such as the box-counting dimension or the set of complex dimensions for a fractal are discussed in [9] and [11], for example.

We will focus on what is known as the Hausdorff dimension. To understand the definition of the Hausdorff dimension of a set, we will first define the notion of Hausdorff measure. Further details can be found in [9].

Definition 2.3 *Let $X \subseteq \mathbb{R}^n$. The **diameter** of a set X is defined by*

$$diam(X) = \sup\{|x-y| : x, y \in X\}.$$

Let $s \geq 0$, and $\delta > 0$. Define

$$\mathcal{H}^s_\delta(X) = \inf\left\{\sum_{i=1}^{\infty} diam(U_i)^s : X \subseteq \bigcup_{i=1}^{\infty} U_i \text{ and } diam(U_i) < \delta\right\}. \tag{1}$$

As δ decreases, the infimum is taken over a smaller number of permissible covers of X so $\mathcal{H}^s_\delta(X)$ will increases. This means the limit

$$\mathcal{H}^s(X) = \lim_{\delta\to 0} \mathcal{H}^s_\delta(X)$$

*is well defined. We call $\mathcal{H}^s(X)$ the **s-dimensional Hausdorff measure of** X. The **Hausdorff dimension** of X is defined to be*

$$\dim_H X = \inf\{s \geq 0 : \mathcal{H}^s(X) = 0\}.$$

To calculate the Hausdorff dimension, we will make use of a theorem from [9] that applies to sets arising from an IFS and which satisfy a certain separation property called the *open set condition*. An IFS, $\{g_i : i = 1, 2, \ldots, k\}$, satisfies the open set condition if there exists a non-empty bounded open set V such that

$$\bigcup_{i=1}^{k} g_i(V) \subset V$$

where this union is disjoint. The Sierpinski gasket and all of the variations on the Sierpinski gasket discussed in this work satisfy the open set condition with V taken to be the interior of the triangle formed by the points p_1, p_2, and p_3. We will therefore use the following theorem when calculating the Hausdorff dimension of sets arising from an IFS which satisfies the open set condition.

Theorem 2.4 *Let g_i be similarities on $\mathbb{R}^N$ with contracting ratios r_i $(1 \leq i \leq k)$ and which satisfy the open set condition. If X is the set satisfying*

$$X = \bigcup_{i=1}^{k} g_i(X)$$

then the Hausdorff dimension of X *is* $\dim_H(X) = s$ *where* s *is the real number satisfying*

$$\sum_{i=1}^{k} r_i^s = 1.$$

The Hausdorff dimension of the Sierpinski gasket for example is $\dim_H(SG) = s = \log_2(3)$ which solves the equation

$$\sum_{i=1}^{3} (1/2)^s = 1.$$

3 An IFS that generates SSG

We first acquaint ourselves with the IFS given in [7] that generates the stretched Sierpinski gasket, SSG. Fix the following points in $\mathbb{R}^2$:

$$p_1 = (0,0) \qquad p_2 = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) \qquad p_3 = (1,0)$$

$$p_4 = \frac{p_2 + p_3}{2} \qquad p_5 = \frac{p_1 + p_3}{2} \qquad p_6 = \frac{p_1 + p_2}{2}.$$

Next fix $\alpha \in (0, \frac{1}{3})$ and for $i = 1, 2, \ldots, 6$ define $G_{\alpha,i} : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$G_{\alpha,i}(x) = A_i(x - p_i) + p_i$$

where

$$A_1 = A_2 = A_3 = \frac{1-\alpha}{2} I = \frac{1-\alpha}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$A_4 = \frac{\alpha}{4} \begin{pmatrix} 1 & -\sqrt{3} \\ -\sqrt{3} & 3 \end{pmatrix} \qquad A_5 = \alpha \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad A_6 = \frac{\alpha}{4} \begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & 3 \end{pmatrix}.$$

The matrices A_1, A_2, A_3 are shrinking a shape by a factor of $\frac{1-\alpha}{2}$ in all directions. This means $G_{\alpha,1}, G_{\alpha,2}, G_{\alpha,3}$ are similarities, and in fact these functions correspond to (i.e. map to) the triangles in SSG. The matrices A_4, A_5, and A_6 will correspond to contractions that map onto the stretched line segments in SSG. These matrices will not be similarities, since they will collapse inputs onto a line segment. We now recall how one can project onto a fixed vector.

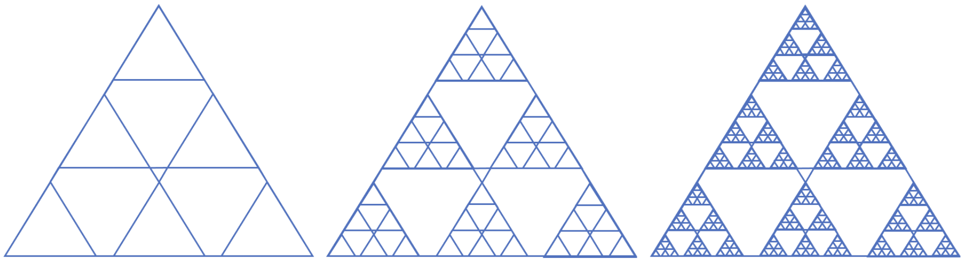

Figure 4: Stages 1, 2, and 3 in the construction of SSG^3.

3.1 Vector projections

Definition 3.1 *Let $v \in \mathbb{R}^N$ be a fixed vector and $s \in \mathbb{R}^N$. The vector projection of s onto a line parallel to v is given by*

$$proj_v(s) = \frac{s \cdot v}{\|v\|^2} v$$

where $s \cdot v$ is the dot product of s and v, and $\|v\| = \sqrt{v_1^2 + v_2^2 + \cdots + v_N^2}$ is the usual magnitude of the vector $v = (v_1, v_2, \ldots, v_N)$.

Vector projections are how we arrive at the matrices A_4, A_5, and A_6. These matrices need to shrink a shape and project onto the stretched line segments in SSG. The matrix A_4 corresponds to projection onto the vector $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$; the matrix A_5 corresponds to projection onto $(1, 0)$; and A_6 corresponds to projection onto $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$. We show below the calculation of A_6 and leave the others as an exercise for the reader.

Let $s = (x, y) \in \mathbb{R}^2$ and $v = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$. Note that $\|v\| = 1$. Then,

$$\begin{aligned}
\mathrm{proj}_v(s) &= \frac{s \cdot v}{\|v\|^2} v \\
&= \left(\frac{1}{2}x + \frac{\sqrt{3}}{2}y\right)\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) \\
&= \frac{1}{4}\left(x + \sqrt{3}y, \sqrt{3}x + 3y\right) \\
&= \frac{1}{4}\begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.
\end{aligned}$$

This gives us the matrix $\frac{1}{4}\begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & 3 \end{pmatrix}$ which will project the vector $s = (x, y)$ onto a line parallel to $v = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$. We then scale this by α to get A_6.

3.2 Fixed points and contractions

Next let's examine the significance of the points $p_1, p_2, \dots p_6$. These are the fixed points of the maps $G_{\alpha,i}$, meaning that for each $i = 1, 2, \dots, 6$ we have

$$G_{\alpha,i}(p_i) = p_i.$$

In fact, there is an important connection between contraction maps and fixed points.

Theorem 3.2 ***Contraction Mapping Principal (Fixed Point Theorem)***
Let $D \subseteq \mathbb{R}^N$ be a closed set. If $f : D \to D$ is a contraction map, then f has a unique fixed point. In other words, there is exactly one $x_0 \in D$ which satisfies

$$f(x_0) = x_0.$$

The fixed points for the maps in an IFS help us shift the scaled and possibly projected vectors resulting from applying A_i into the correct positions. For example, in the contraction

$$G_{\alpha,2}(x) = A_2(x - p_2) + p_2$$

the matrix A_2 will scale the outer triangle formed by p_1, p_2 and p_3 into a smaller triangle. The point p_2 will shift this scaled triangle into the correct position.

We can now recognize how each of the functions $G_{\alpha,i}$ corresponds to some component of the stretched Sierpinski gasket. Namely, a triangle for $G_{\alpha,1}, G_{\alpha,2}$, and $G_{\alpha,3}$ and line segments for $G_{\alpha,4}, G_{\alpha,5}$, and $G_{\alpha,6}$. Reviewing the IFS for the stretched Sierpinski gasket gives us an idea of where to begin to build the IFS for SSG^n. Each triangle and each line segment in SSG^n will correspond to a map in the IFS and hence a fixed point and projection/scaling matrix.

4 An IFS for SSG^3

Our next step is to construct the contractions that will generate SSG^n. For simplicity, we initially focus on $n = 3$. For each contraction we must determine a fixed point and a projection/scaling matrix. We will need 6 maps for the triangles in SSG^3 and 9 maps for the line segments in SSG^3; see Figures 4 and 5.

4.1 Fixed points for SSG^3

We determine that the following points are fixed points for the maps in the IFS generating the 6 triangles in the construction of SSG^3:

$$s_1 = p_1 = (0,0) \qquad s_2 = p_2 = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) \qquad s_3 = p_3 = (1,0)$$

$$s_4 = \frac{p_1 + p_2}{2} \qquad s_5 = \frac{p_2 + p_3}{2} \qquad s_6 = \frac{p_1 + p_3}{2}.$$

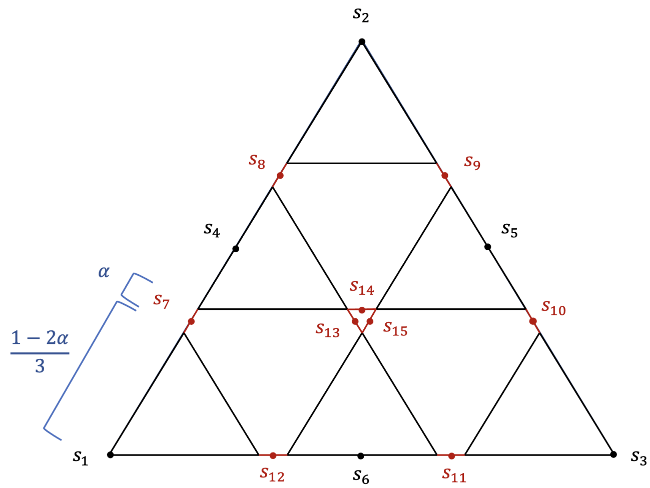

Figure 5: Stage 1 in the construction of SSG^3 with the fixed points s_i.

The reader should note the renaming and shuffling of the points p_4, p_5, p_6 to the points s_4, s_5, s_6. This is done for convenience moving forward. Calculating the fixed points for the maps in the IFS which would correspond to the line segments in SSG^3 is our next task.

To generate the points s_7 and s_8 as seen in Figure 5, we scale the point $s_2 = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$. We arrived at

$$s_7 = \left(\frac{1-2\alpha}{3} + \frac{\alpha}{2}\right) s_2 \qquad \text{and} \qquad s_8 = \left(\frac{2-4\alpha}{3} + \frac{3\alpha}{2}\right) s_2$$

where α is again a real number that represents the length by which the level-3 Sierpinski gasket is stretched. The scaling factors above will be denoted by

$$\alpha_1 = \left(\frac{1-2\alpha}{3} + \frac{\alpha}{2}\right) \qquad \text{and} \qquad \alpha_2 = \left(\frac{2-4\alpha}{3} + \frac{3\alpha}{2}\right)$$

In a similar manner and using the symmetry of SSG^3 we generate the following for the points s_9, s_{10}, s_{11}, and s_{12}:

$$\begin{aligned} s_9 &= s_2 \begin{pmatrix} -\alpha_2 & 0 \\ 0 & \alpha_2 \end{pmatrix} + (1, 0) \\ s_{10} &= s_2 \begin{pmatrix} -\alpha_1 & 0 \\ 0 & \alpha_1 \end{pmatrix} + (1, 0) \\ s_{11} &= \alpha_2 s_3 \\ s_{12} &= \alpha_1 s_3 \end{aligned}$$

We now calculate the fixed points on the "inside" of SSG^3, namely s_{13}, s_{14}, and s_{15}. Again we will make use of the symmetry of SSG^3 as well as our knowledge of the points s_2 and s_7.

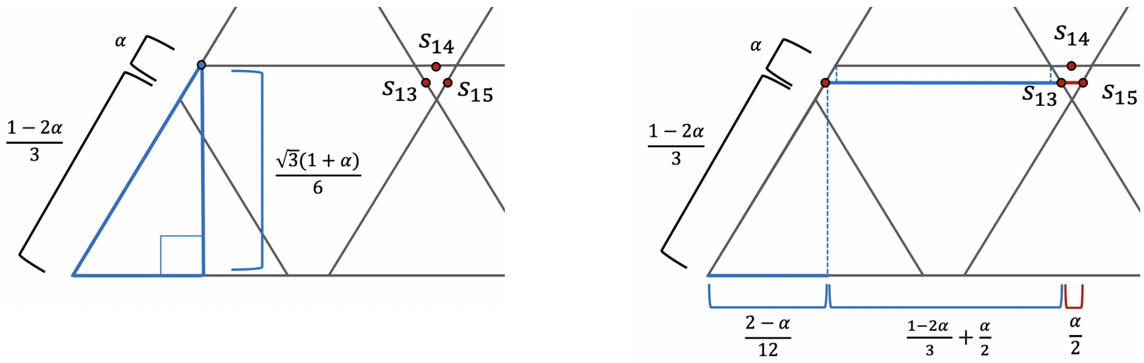

Figure 6: Components of the calculation to find s_{14} (left), s_{13} and s_{15} (right).

Calculation of s_{14}

Note that the x-coordinate of s_{14} is the same as that of s_2. It remains to calculate the y-coordinate of s_{14}. For this we note the right triangle in Figure 6 (left). Since the initial outer triangle is an equilateral triangle, we know the angles within the outer triangle are $60°$. A trigonometry calculation yields

$$\frac{\sqrt{3}(1+\alpha)}{6}$$

for the y-coordinate of s_{14}. This gives us

$$s_{14} = s_2 \begin{pmatrix} 1 & 0 \\ 0 & \frac{1+\alpha}{3} \end{pmatrix}$$

Calculation of s_{13} and s_{15}

The y-coordinate for both s_{13} and s_{15} is the same as that of s_7. We now calculate the x-coordinate for both s_{13} and s_{15}. Again a trigonometric calculation shows that the x-coordinate for s_{13} is given by

$$\frac{2-\alpha}{12} + \frac{1-2\alpha}{3} + \frac{\alpha}{2} \quad = \quad \frac{2-\alpha}{4}.$$

This gives us

$$s_{13} = s_2 \begin{pmatrix} \frac{2-\alpha}{2} & 0 \\ 0 & \frac{2-\alpha}{6} \end{pmatrix}$$

For the x-coordinate of s_{15} we add an additional length of $\frac{\alpha}{2}$ to the x-coordinate of s_{13}. This gives

$$s_{15} = s_2 \begin{pmatrix} \frac{2+\alpha}{2} & 0 \\ 0 & \frac{2-\alpha}{6} \end{pmatrix}$$

4.2 Projection/scaling matrices for SSG^3

Next we need a matrix that simply scales the outer triangle to the size of one of the small triangles shown in Figure 5. We define matrices X_i where $i = 1, 2, \ldots, 6$ by

$$X_i = \frac{1-2\alpha}{3} I.$$

Next we need matrices that scale and project onto the stretched line segments in SSG^3. First define

$$B = \frac{\alpha}{4}\begin{pmatrix} 1 & -\sqrt{3} \\ -\sqrt{3} & 3 \end{pmatrix} \qquad C = \alpha\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad D = \frac{\alpha}{4}\begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & 3 \end{pmatrix}.$$

The reader will recognize these as the projection and scaling matrices A_4, A_5, A_6 from the definition of SSG. Define matrices X_i for $i = 7, 8, \ldots 15$ as follows:

$$\begin{aligned} X_7 = X_8 = X_{15} &= D \\ X_9 = X_{10} = X_{13} &= B \\ X_{11} = X_{12} = X_{14} &= C. \end{aligned}$$

Note that these are grouped according to which line segments are parallel in SSG^3. We are now able to state the IFS that generates SSG^3.

Definition 4.1 *An IFS which generates SSG^3 is given by*

$$H_{\alpha,i}(x) = X_i(x - s_i) + s_i$$

where s_i and X_i for $i = 1, 2, \ldots, 15$ are as defined previously.

4.3 An IFS for SSG^n

We can extend the construction above to SSG^n for $n \geq 4$. First note that there are

$$N = \frac{n(n+1)}{2}$$

upright triangles in SSG^n. Of these there are

$$\frac{(n-2)(n-3)}{2}$$

upright triangles in the interior of the set. First, we categorize the fixed points and scaling matrices of SSG^n into the following four groups:

Group 1 Boundary and α-independent. These are the fixed points associated to the triangles along the edges of the set. There are

$$\frac{n(n+1)}{2} - \frac{(n-2)(n-3)}{2} = 3n - 3$$

of these triangles so our first $3n - 3$ fixed points will be associated to these.

Group 2 Interior and α-independent. These are the fixed points associated to the interior upright triangles in SSG^n. The next

$$\frac{(n-2)(n-3)}{2}$$

fixed points will be associated to these interior triangles.

Group 3 Boundary and α-dependent. These will correspond to the stretched line segments along the edges of SSG^n. There are

$$3(n-1)$$

of these fixed points.

Group 4 Interior and α-dependent. These correspond to the stretched line segments that appear in the interior of SSG^n. We have

$$\frac{3}{2}(n-1)(n-2)$$

such fixed point. All together SSG^n contains $n(2n-1)$ fixed points.

Note that the fixed points from Groups 1 and 2 above are indeed the fixed points from SG^n. These are left unchanged by the stretching of the Sierpinski gasket. Each fixed point in the first two groups corresponds to scaling matrix A where,

$$A = \frac{1-(n-1)\alpha}{n} I.$$

The fixed points in Group 3–Boundary and α-Dependent–can be acquired in the following manner. For each $n \geq 4$, we define the following set of constants $\{\alpha_k\}_{k=1}^{n-1}$ where,

$$\alpha_k = k\left(\frac{1-(n-1)\alpha}{n}\right) + (2k-1)\frac{\alpha}{2}.$$

The reader may recognize that within SSG^n, the upright triangles have side lengths of $\frac{1-(n-1)\alpha}{n}$ and $\frac{\alpha}{2}$ is half the length of the boundary line segments between the triangles.

Note that the index for the first point in group 3 will be $i = N+1$. For fixed points along the left edge of the outer triangle in SSG^n, we use

$$s_i = \alpha_{(i-N)} s_2$$

with scaling matrix D. The points along the right edge of the outer triangle for SSG^n will begin with index $i = N + (n-1) + 1$. These can be acquired by

$$s_i = (-1, 1)s_{i-(n-1)} + s_3.$$

with scaling matrix B. Finally, for the points along the bottom edge of the outer triangle we use,

$$s_i = \alpha_{i-(N+2(n-1))} s_3$$

where the first index is $i = N + 2(n-1) + 1$, the last fixed point at $i = N + 3(n-1)$, and associated scaling matrix C. For simple notation, We set $N' = N + 3(n-1)$.

Lastly for the three point clusters, notice they lie along the midpoints of the downward triangles of side length α. Moreover, the centers of the α triangles are the vertices that appear in the interior of the regular level-n Sierpinski gasket. For SSG^n we will have

$$m = \frac{(n-1)(n-2)}{2}$$

centers. Call these $c_1, c_2, \ldots, c_m$. Each center will give a cluster of 3 fixed points for SSG^n. So the first center c_1 corresponds to the fixed points with indices,

$$i = N' + 1,\ N' + 2,\ N' + 3$$

and center c_p corresponds to indices,

$$i = N' + 3(p-1) + 1,\ N' + 3(p-1) + 2,\ N' + 3(p-1) + 3$$

The fixed points can be obtained by the following formula:

$$s_i = c_p + \frac{\alpha\sqrt{3}}{6\|c_p\|} R_{(i-(N'+3(p-1)))} c_p^T.$$

where $R_k = \begin{pmatrix} \cos\theta_k & -\sin\theta_k \\ \sin\theta_k & \cos\theta_k \end{pmatrix}$ for $\theta_1 = 90^\circ - \arctan\left(\frac{y_{c_p}}{x_{c_p}}\right)$, $\theta_2 = 120^\circ + \theta_1$, $\theta_3 = 240^\circ + \theta_1$. For the fixed points corresponding to θ_1, θ_2, θ_3, the respective scaling matrices are C, B, and D. At last we construct an IFS for SSG^n.

Definition 4.2 *An IFS which generates SSG^n is given by*

$$H_{\alpha,n,i}(x) = X_i(x - s_i) + s_i$$

where s_i and X_i for $i = 1, 2, \ldots, n(2n-1)$ are defined above.

Figure 7 illustrates the usefulness of the IFS given above. Starting with the fixed points in the IFS, one can successively apply the maps in the IFS to generate additional points on SSG^n. This gives us a method to approximate sets like SSG^4 and SSG^5 seen in Figure 7.

5 The Hausdorff dimension of SSG^n

We now need one additional theorem regarding the Hausdorff dimension. This theorem and its proof can be found in [9].

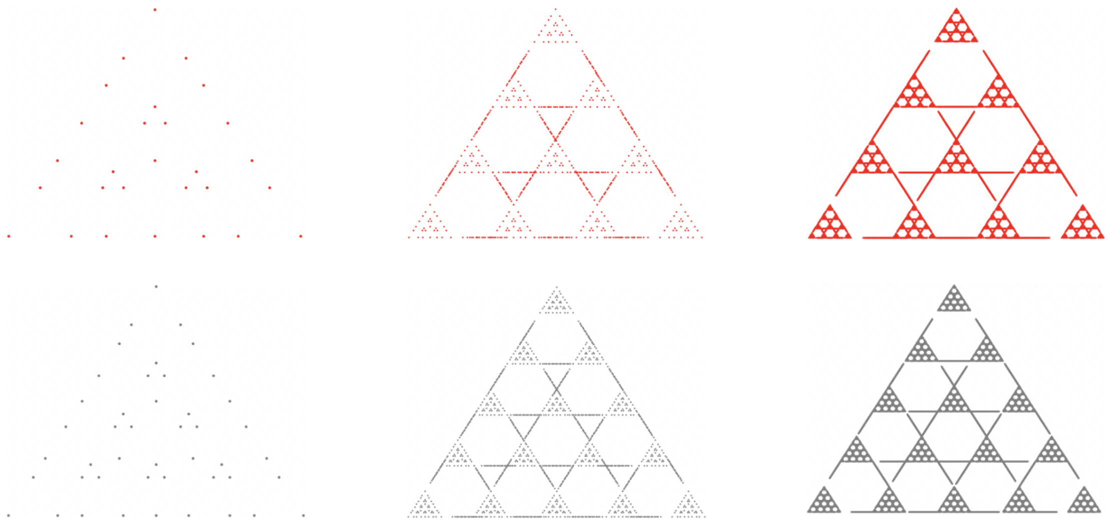

Figure 7: The fixed points for SSG^4 and SSG^5 (left) are shown above along with a first (center) and second (right) application of the IFS maps to the set of fixed points.

Theorem 5.1 *The Hausdorff dimension is countably stable:*

$$\dim_H\left(\bigcup_{i\in\mathbb{N}} U_i\right) = \sup_{i\in\mathbb{N}} \dim_H(U_i).$$

To make use of this theorem, we observe the following useful decomposition of SSG^n. Let F^n be the unique compact set generated by the IFS which consists of only the similarity maps in the IFS for SSG^n. More specifically, F^n is the union of all triangles in SSG^n. Next, let J^n be the union of all open (excluding end points) line segments which form the stretched portion of SSG^n. One can immediately extend a result in [7] (Lemma 2.1.1) to get that for any $n \geq 2$, we have

$$SSG^n = F^n \bigcup J^n$$

where the union is disjoint. We now arrive at the Hausdorff dimension of SSG^n.

Theorem 5.2 *For $n \geq 2$ and $\alpha \in \left(0, \frac{1}{n+1}\right)$, the Hausdorff dimension of SSG^n is*

$$\dim_H SSG^n = \frac{\ln\left(\frac{n(n+1)}{2}\right)}{\ln(n) - \ln(1-(n-1)\alpha)}.$$

Proof. By the countable stability of the Hausdorff dimension, we know

$$\dim_H SSG^n = \sup\left\{\dim_H F^n, \dim_H J^n\right\}.$$

Notice that F^n is the unique attractor associated to an IFS with $N = \frac{n(n+1)}{2}$ similarities, each of ratio $r = \frac{1-(n-1)\alpha}{n}$. Then by Theorem 2.4, there exists a unique $s > 0$ such that,

$$\sum_{i=1}^{N} r^s = \sum_{i=1}^{N} \left(\frac{1-(n-1)\alpha}{n}\right)^s = 1$$

and $s = \dim_H F^n$. Solving for s gives

$$\frac{n(n+1)}{2}\left(\frac{1-(n-1)\alpha}{n}\right)^s = 1 \tag{2}$$

$$s \cdot \ln\left(\frac{1-(n-1)\alpha}{n}\right) = -\ln\left(\frac{n(n+1)}{2}\right) \tag{3}$$

$$s = \frac{-\ln\left(\frac{n(n+1)}{2}\right)}{\ln\left(\frac{1-(n-1)\alpha}{n}\right)} \tag{4}$$

$$s = \frac{\ln\left(\frac{n(n+1)}{2}\right)}{\ln(n) - \ln(1-(n-1)\alpha)}. \tag{5}$$

Note that any line segment has Hausdorff dimension 1. So for J^n, a countable union of line segments, we have

$$\dim_H J^n = 1.$$

This now gives us

$$\dim_H SSG^n = \max\left\{\frac{\ln\left(\frac{n(n+1)}{2}\right)}{\ln(n) - \ln(1-(n-1)\alpha)}, 1\right\}.$$

Notice that $n \geq 2$ and $\alpha \in (0, \frac{1}{n+1})$ imply that

$$\frac{\ln\left(\frac{n(n+1)}{2}\right)}{\ln(n) - \ln(1-(n-1)\alpha)} > 1.$$

Thus, for a fixed $\alpha \in (0, \frac{1}{n+1})$ where $n \geq 2$, the Hausdorff dimension of SSG^n is given by

$$\dim_H SSG^n = \frac{\ln\left(\frac{n(n+1)}{2}\right)}{\ln(n) - \ln(1-(n-1)\alpha)}.$$

$\square$

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