A Closed Structure Constant Formula for the Universal Enveloping Algebra of the Lie Algebra $\mathfrak{sl}_2$

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Abstract - Recently, Gourley and Kennedy gave a recursive formula for the structure constants of the universal enveloping algebra of $\mathfrak{sl}_2$. Using Kostant's formula we give a closed formula for these structure constants.

Keywords : Lie algebras; structure constants; universal enveloping algebras

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1 Introduction

We will give a brief overview of the necessary background. Details can be found in [3].

Let $\mathbb{C}$ denote the field of complex numbers, $\mathbb{N}$ denote the positive integers and $\mathbb{Z}_+$ denote the non-negative integers. $\mathfrak{sl}_2$ is the vector space over of all 2×2 matrices with complex entries and trace 0 (the trace of a matrix is the sum of the entries on the main diagonal) equipped with the commutator bracket; $[A, B] = AB - BA$ for any $A, B \in \mathfrak{sl}_2$. The standard ordered basis for $\mathfrak{sl}_2$ is made up of the following elements:

$$f := \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},\ h := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},\ e := \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Given any Lie algebra L denote its universal enveloping algebra by $U(L)$. $U(L)$ is the unique unital associative algebra over $\mathbb{C}$ such that there is a linear function $\iota : L \to U(L)$ such that

$$\iota([A, B]) = \iota(A)\iota(B) - \iota(B)\iota(A) \tag{1}$$

for all $A, B \in L$, and the following holds: for any unital associative algebra, V, over $\mathbb{K}$ and any linear function $j : L \to V$ satisfying (1), there exists a unique algebra homomorphism $\phi : U(L) \to V$ such that $\phi(1) = 1$ and $\phi \circ \iota = j$. Informally $U(L)$ is made up of all linear combinations over $\mathbb{C}$ of all formal products of the basis elements of L in any order. Also, for all $A, B \in L$, $AB = BA + [A, B]$ in $U(L)$. In general, $U(L)$ is not commutative.

Given any algebra A, over any field, with basis $B = \{e_i\}$ the structure constants of A with respect to the basis B are the scalars $c_{i,j}^k$ such that

$$e_i e_j = \sum_k c_{i,j}^k e_k.$$

We will find the structure constants for the algebra $U(\mathfrak{sl}_2)$.

Given any Lie algebra L over $\mathbb{C}$, the Poincaré-Birkhoff-Witt (PBW) Theorem allows us to uniquely write the products in $U(L)$ as linear combinations of products where the basis elements of L are in any order we choose.

Theorem 1.1 ((PBW) Theorem) *Let L be a Lie algebra with basis $\{x_i \mid i \in I\}$, and let $<$ be an ordering of I. Then*

$$\left\{x_{i_1}^{r_1} \dots x_{i_n}^{r_n} \mid i_1, \dots i_n \in I,\ i_1 < \cdots < i_n,\ r_1, \dots, r_n \in \mathbb{Z}_+\right\}$$

is a basis for $U(L)$.

Given two arbitrary products in the universal enveloping algebra of a Lie algebra, $U(L)$, written using our chosen ordering of the basis of L, we can reorder their product by the PBW Theorem using a fixed basis ordering. The coefficients in the resulting linear combination of products of basis elements in the our chosen order are called the structure constants. The PBW Theorem does not tell us how to find these specific structure constants. Finding them is an area of active research.

In [2], Gourley and Kennedy found the structure constants for the Lie algebra $\mathfrak{sl}_2$. Their formula is recursive and doesn't make use of Kostant's formula for reordering a product of the form $e^r f^t$, for any $r, t \in \mathbb{Z}_+$, appearing in [4].

In this work, we first use Kostant's formula for reordering such a product to give a closed formula for the structure constants of $\mathfrak{sl}_2$ and then we extend this result to find a closed formula for the structure constants for the Lie algebra of type D_2.

2 Using Kostant's Formula to Get a Closed Structure Constant Formula for $\mathfrak{sl}_2$

2.1 Kostant's Formula

In the universal enveloping algebra of the Lie algebra $\mathfrak{sl}_2$, $U(\mathfrak{sl}_2)$, define the kth divided power of an element $u \in U(\mathfrak{sl}_2)$ to be $u^{(k)} = \frac{u^k}{k!}$. Kostant in [4] found a formula for straightening a general product in $U(\mathfrak{sl}_2)$ of the form $e^{(r)} f^{(t)}$, for all $r, t \in \mathbb{Z}^+$ as follows:

$$e^{(r)} f^{(t)} = \sum_{j=0}^{\min\{r,t\}} f^{(t-j)} \binom{h - r - t + 2j}{j} e^{(r-j)}.$$

Multiplying both sides of this equation by $r!t!$ and using appropriate binomial coefficients we get:

$$e^r f^t = \sum_{j=0}^{\min\{r,t\}} \binom{r}{j} \binom{t}{j} (j!)^2 f^{t-j} \binom{h - r - t + 2j}{j} e^{r-j}.$$

We need a version of this formula in which the binomial $\binom{h-r-t+2j}{j}$ has been expanded. This version will involve the Stirling numbers of the first kind. The following information can be found. These numbers are denoted by $s(n,m)$ where $n,m \in \mathbb{Z}_+$ and $(-1)^{n-m}s(n,m)$ is the number of permutations of n symbols which have exactly m cycles according to Section 24.1.3 of [1]. Therefore, $s(n,m)=0$ if $m>n$. Also,

$$\binom{x}{n} = \sum_{m=0}^{n} s(n,m)x^m.$$

Using this formula we get, for all $r,t \in \mathbb{Z}_+$:

$$\begin{aligned}
e^r f^t &= \sum_{j=0}^{\min\{r,t\}} \binom{r}{j}\binom{t}{j}(j!)^2 f^{t-j}\binom{h-r-t+2j}{j}e^{r-j} \\
&= \sum_{j=0}^{\min\{r,t\}} \binom{r}{j}\binom{t}{j}(j!)^2 f^{t-j}\frac{1}{j!}(h-r-t+2j)(h-r-t+2j-1) \\
&\cdots(h-r-t+2j-j+1)e^{r-j} \\
&= \sum_{j=0}^{\min\{r,t\}} \binom{r}{j}\binom{t}{j} j! f^{t-j}\sum_{k=0}^{j} s(j,k)(h-r-t+2j)^k e^{r-j}
\end{aligned}$$

Where $s(j,k)$ is a Stirling number of the first kind, see [1] for details.

$$\begin{aligned}
&= \sum_{j=0}^{\min\{r,t\}}\sum_{k=0}^{j} \binom{r}{j}\binom{t}{j} j! s(j,k) f^{t-j}(h+2j-r-t)^k e^{r-j} \\
&= \sum_{j=0}^{\min\{r,t\}}\sum_{k=0}^{j} \binom{r}{j}\binom{t}{j} j! s(j,k) f^{t-j}\sum_{l=0}^{k}\binom{k}{l} h^{k-l}(2j-r-t)^l e^{r-j} \\
&= \sum_{j=0}^{\min\{r,t\}}\sum_{k=0}^{j}\sum_{l=0}^{k} \binom{r}{j}\binom{t}{j}\binom{k}{l} j! s(j,k)(2j-r-t)^l f^{t-j}h^{k-l}e^{r-j}
\end{aligned}$$

Note that this gives the closed formula:

$$e^r f^t = \sum_{j=0}^{\min\{r,t\}}\sum_{k=0}^{j}\sum_{l=0}^{k} \binom{r}{j}\binom{t}{j}\binom{k}{l} j! s(j,k)(2j-r-t)^l f^{t-j}h^{k-l}e^{r-j} \qquad (2)$$

where the terms on the right-hand side have the basis elements f, h, and e in the preferred order.

2.2 A Closed Formula for Gourley and Kennedy's Structure Constants

In [2], the authors gave a recursive formula to straighten $e^r f^t$. This then led to a recursive formula for the structure constants of $U(\mathfrak{sl}_2)$ in which they straightened a product of the form $(f^p h^q e^r)(f^v h^t e^u)$ for any $p,q,r,v,t,u \in \mathbb{Z}_+$. For our closed formula for these structure constants we used (2) and the following formulas from Proposition 7 in [2]:

$$e^t h^v = (h-2t)^v e^t = \sum_{k=0}^{v} \binom{v}{k} (-2t)^k h^{v-k} e^t; \tag{3}$$

$$h^s f^u = f^u (h-2u)^s = \sum_{k=0}^{u} \binom{u}{k} (-2u)^k f^u h^{u-k}. \tag{4}$$

We can now state and prove our closed formula.

Theorem 2.1 *For all $p, q, r, t, u, v \in \mathbb{Z}_+$ we have:*

$$\left(f^p h^q e^r\right)\left(f^v h^t e^u\right) = \sum_{j=0}^{\min\{r,v\}} \sum_{k=0}^{j} \sum_{l=0}^{k} \sum_{m=0}^{q} \sum_{n=0}^{t} \binom{r}{j}\binom{v}{j}\binom{k}{l}\binom{q}{m}\binom{t}{n} j! s(j,k)$$
$$\times \left(2j-r-v\right)^l \left(-2\right)^{m+n} \left(v-j\right)^m \left(r-j\right)^n f^{p+v-j} h^{q+t+k-(m+n+l)} e^{u+r-j}.$$

Proof.

$$\begin{aligned}
&\left(f^p h^q e^r\right)\left(f^v h^t e^u\right)\\
&= f^p h^q \left(e^r f^v\right) h^t e^u\\
&= f^p h^q \left(\sum_{j=0}^{\min\{r,v\}} \sum_{k=0}^{j} \sum_{l=0}^{k} \binom{r}{j}\binom{v}{j}\binom{k}{l} j! s(j,k)(2j-r-v)^l f^{v-j} h^{k-l} e^{r-j}\right) h^t e^u \text{ by (2)}\\
&= f^p h^q \left(\sum_{j=0}^{\min\{r,v\}} \sum_{k=0}^{j} \sum_{l=0}^{k} \binom{r}{j}\binom{v}{j}\binom{k}{l} j! s(j,k)(2j-r-v)^l f^{v-j} h^{k-l} \left(e^{r-j} h^t\right)\right) e^u\\
&= f^p h^q \Bigg(\sum_{j=0}^{\min\{r,v\}} \sum_{k=0}^{j} \sum_{l=0}^{k} \binom{r}{j}\binom{v}{j}\binom{k}{l} j! s(j,k)(2j-r-v)^l f^{v-j} h^{k-l}\\
&\times \left(\left(h-2\left(r-j\right)\right)^t e^{r-j}\right)\Bigg) e^u \text{ by (3)}\\
&= f^p \sum_{j=0}^{\min\{r,v\}} \sum_{k=0}^{j} \sum_{l=0}^{k} \binom{r}{j}\binom{v}{j}\binom{k}{l} j! s(j,k)(2j-r-v)^l \left(h^q f^{v-j}\right) h^{k-l}\\
&\times \left(\left(h-2\left(r-j\right)\right)^t e^{u+r-j}\right)\\
&= f^p \sum_{j=0}^{\min\{r,v\}} \sum_{k=0}^{j} \sum_{l=0}^{k} \binom{r}{j}\binom{v}{j}\binom{k}{l} j! s(j,k)(2j-r-v)^l f^{v-j} \left(h-2\left(v-j\right)\right)^q h^{k-l}\\
&\times \left(\left(h-2\left(r-j\right)\right)^t e^{u+r-j} \text{ by (4)}
\end{aligned}$$

$$= \sum_{j=0}^{\min\{r,v\}} \sum_{k=0}^{j} \sum_{l=0}^{k} \binom{r}{j}\binom{v}{j}\binom{k}{l} j! s(j,k)(2j-r-v)^l f^{p+v-j} \left(h - 2\left(v-j\right)\right)^q h^{k-l}$$
$$\times \left(\left(h - 2\left(r-j\right)\right)^t e^{u+r-j}\right.$$
$$= \sum_{j=0}^{\min\{r,v\}} \sum_{k=0}^{j} \sum_{l=0}^{k} \sum_{m=0}^{q} \sum_{n=0}^{t} \binom{r}{j}\binom{v}{j}\binom{k}{l}\binom{q}{m}\binom{t}{n} j! s(j,k)(2j-r-v)^l \left(-2\right)^{m+n}$$
$$\times \left(v-j\right)^m \left(r-j\right)^n f^{p+v-j} h^{q+t+k-(m+n+l)} e^{u+r-j} \text{ by (3) and (4).}$$

□

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