Mountain Counting

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Abstract - We investigate the combinatorial objects, mountains of spheres, which are three-dimensional variants of two-dimensional fountains of coins. A mountain can be decomposed into a sequence of fountains, and this decomposition leads to a recurrence relation. We obtain a closed-form solution for their counts, based on this recurrence relation. Lastly, we discuss the feasibility of this enumeration.

Keywords : enumerative combinatorics; generating functions

Mathematics Subject Classification (2020) **:** 05A15

1 Introduction

A common problem in combinatorics is to enumerate the ways to piece together physical or mathematical objects that are subject to certain rules. For example, a *polyomino* is a collection of unit squares in a square lattice subject to the rule that the squares must be edge-wise connected, i.e., for any pair of squares in the polyomino there is a step-by-step path of squares totally within the polyomino, beginning and ending at the pair, that share an edge from one step to the next. Martin Gardner in his columns in *Scientific American* popularized polyominoes in the 1960's [3]. The literature about them is vast, but popular overviews can be found in [4] and [6]. Figure 1 displays four polyominoes. There are several ways to consider polyominoes equivalent, by translation, rotation, and reflection, to name a few. Despite tremendous effort, closed-form formulas for the number of polyominoes in terms of their number of squares are not known [10].

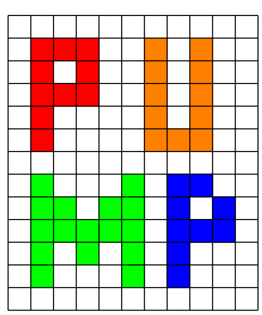


Figure 1: Polyominoes in a square lattice

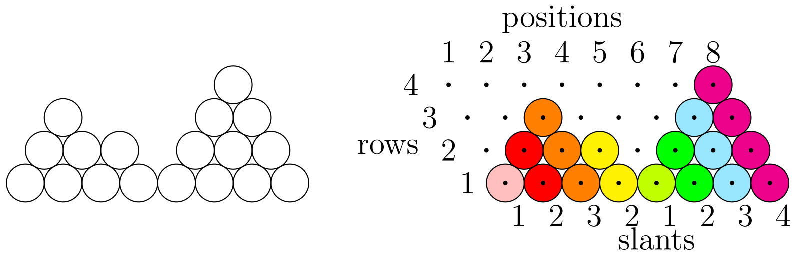


Figure 2: A fountain of coins of width 8 and the same fountain with its slant representation $\langle 1,2,3,2,1,2,3,4\rangle$, embedded in a coordinate system of rows and positions.

Another interesting combinatorial object is a *fountain of coins*. An (n, w) fountain of coins is an arrangement of n identical coins subject to two rules: the bottom row must consist of w contiguous coins, and the coins in higher rows must satisfy the *gravity rule* that each coin has to rest on two coins in the row immediately below. The number of coins in the bottom row of a fountain is called its *width*. Figure 2 shows an $(18, 8)$ fountain of width 8, consisting of 18 total coins. Odlyzko and Wilf analyzed fountains of coins in [7] and attributed the question of their enumeration to J.M. Propp. They noted that the number of fountains with exactly w coins in the bottom row coincides with the w^{th} Catalan number, which we denote $C_w = \binom{2w}{w}/(w+1) \approx 4^w/w^{3/2}\sqrt{\pi}$ (see [8]). More significantly, they showed that if a_n denotes the number of fountains with n total coins, then the generating function $f(x) = \sum_n a_n x^n$ is the continued fraction

$$f(x) = \cfrac{1}{1-\cfrac{x}{1-\cfrac{x^2}{1-\cfrac{x^3}{1-\cdots}}}} \quad . \tag{1}$$

This continued fraction was first studied by S. Ramanujan, and an identity in his "lost notebook" [1] provided a direct method in [7] to approximate the coefficients of this generating function.

The combinatorial objects of interest in this paper are *mountains of spheres*, which are three-dimensional variants of fountains of coins. Mountains of spheres were first introduced by Endicott et al. in [2]. As defined there, an (n, w, ℓ) *mountain* is an arrangement of n identical spheres into a three-dimensional lattice that follows two rules: its bottom layer forms a $w \times \ell$ rectangular grid of spheres, and higher layers follow the gravity rule that every sphere must rest on four spheres in the layer immediately below. Figure 3 shows a $(153, 9, 10)$ mountain. In general, a mountain can have many peaks, valleys, and plateaus. The spheres in each horizontal layer of a mountain must follow the gravity rule, but are otherwise arbitrary and resemble collections of polyominoes. In [2], Endicott analyzed special classes of mountains, such as pyramidal mountains which have a single spire consisting of rectangular cross sections. Here, we consider general mountains and

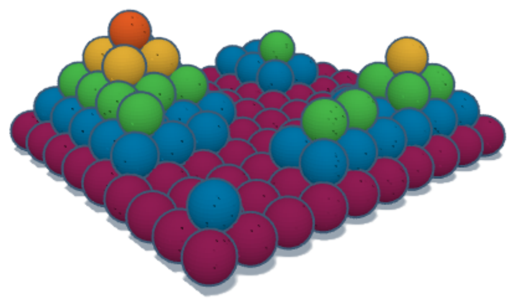


Figure 3: A $(153, 9, 10)$ mountain, colored by horizontal layers

enumerate them in three ways:

$$m_{n,w,\ell} = \text{the number of mountains consisting of } n \text{ spheres with a } w \times \ell \text{ base;}$$
$$m_{w,\ell} = \sum_{n} m_{n,w,\ell} = \text{the number of mountains with a } w \times \ell \text{ base;}$$
$$m_n = \sum_{w,\ell} m_{n,w,\ell} = \text{the number of mountains consisting of } n \text{ spheres.}$$

Rotations and reflections in general produce different mountains but with the same number of spheres. So, $m_{w,\ell} = m_{\ell,w}$.

The goal of this paper is to generalize the results of [7] from two-dimensional fountains to three-dimensional mountains, and in doing so, answer open questions about the enumeration of mountains from [2]. In Section 2, we review basic results about fountains of coins, as they apply to mountains. Then, in Section 3, we show how a mountain can be sliced face-by-face into fountains, how these faces must be related, and how this relation leads to a linear recurrence relation for $m_{n,w,\ell}$. We then use this recurrence relation to obtain a closed-form formula for $m_{w,\ell}$ in terms of the powers of a matrix. We conclude in Section 4 with remarks on the computational complexity of this solution and mention several open problems about mountain counting.

2 Fountains

Recall that a fountain is an arrangement of identical coins into rows that has a contiguous bottom row and satisfies the two-dimensional gravity rule. More formally, we can also represent a fountain as a collection of ordered pairs of positive integers as follows. Each coin in a fountain is uniquely identified by the row r that it is in and its position k within this row, i.e., by the pair (r, k). Figure 2 illustrates the coordinate system of rows and positions. Note that the position of a coin in one row is the same as the position of the coin to its right in the row below. So, as an alternate definition in terms of integers, we can say that $F \subseteq \mathbb{N} \times \mathbb{N}$ is a (n, w) fountain if $|F| = n$, the bottom row has $(1, k) \in F$

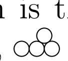
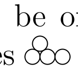
for $1 \leq k \leq w$ and $(1, k) \notin F$ for any other k, and for the gravity rule that if $r > 1$ and $(r, k) \in F$, then $(r-1, k-1)$ and $(r-1, k)$ are also in F.

Though it may be tempting to consider fountains row by row, it often is more fruitful instead to slice them diagonally into *slants*, as illustrated with different colors in Figure 2. We say the k^{th} *slant* of a fountain is the set of all coins with constant position k in the rows. A fountain of width w has w coins on its bottom row and so has w slants, all of which have at least one coin. By the gravity rule, we can see that a slant has no "gaps"—if one coin is present in a slant, then it must be supported by a strip of coins beneath it in its slant. Because of this, any fountain of width w may be expressed completely as a w-tuple of the numbers of coins on its slants as in Figure 2. Notationally, we define $F = \langle s_1, \ldots, s_w \rangle$ to be the fountain with s_k coins on the k^{th} slant for $1 \leq k \leq w$. Then, $|F| = \sum_k s_k$ is the total number of coins in the fountain.

This representation of a fountain as a sequence of integers is much simpler than keeping track of the location of each coin in the fountain individually. In this light, we would like to know exactly when a sequence of positive integers $\langle s_1, \ldots, s_w \rangle$ actually represents a valid fountain. The leftmost slant s_1 must be exactly 1 because there are no coins to the left of the beginning of any fountain. The highest coin in the k^{th} slant has coordinates (s_k, k) and, as long as $s_k > 1$, by the gravity rule must be supported by the coin below and to the left, which has coordinates $(s_k - 1, k - 1)$. Therefore, the number coins s_{k-1} in the $k - 1^{\text{th}}$ slant must be at least $s_k - 1$. So, a sequence $\langle s_1, \ldots, s_w \rangle$ of positive integers represents a fountain if and only if

$$\begin{aligned} s_1 &= 1 \\ s_{k-1} &\geq s_k - 1 \text{ for } 1 < k \leq w. \end{aligned} \tag{2}$$

In other words, the first slant must be exactly 1 and each slant must properly "lean on" the previous slant to the left. Sequences satisfying Equation 2 are one of the sixty six incarnations of combinatorial objects in [11] (example U, page 224) giving rise to the Catalan numbers.

Another advantage of the slant representation is that fountains can be ordered, in particular, with the dictionary order. For instance,  $= \langle 1, 1, 2 \rangle$ precedes  $= \langle 1, 2, 1 \rangle$ in the dictionary order.

Perhaps the most important use for this slant representation is that it provides a recursive method for building fountains. For this recursion, we define the following refinements of types of fountains:

$$\begin{aligned} f_{n,w,s} &= \text{the number of } (n, w) \text{ fountains with } s \text{ coins on the rightmost slant} \\ f_{w,s} &= \sum_n f_{n,w,s} \\ &= \text{the number of fountains of width } w \text{ with } s \text{ coins on the rightmost slant} \\ f_w &= \sum_s f_{w,s} = \text{the number of fountains of width } w \end{aligned}$$

Any fountain can be extended in width by tacking a slant of coins onto the rightmost slant, as limited by Equation 2. Therefore,

$$f_{n,1,s} = \begin{cases} 1 & \text{if } n = s = 1 \\ 0 & \text{otherwise} \end{cases}$$
$$f_{n,w,s} = \sum_{s' \geq s-1} f_{n-s,w-1,s'} \text{ for } n \geq s,\ w > 1,\ s \geq 1. \tag{3}$$

It is easy to compute $f_{w,s}$ by summing over n and to compute f_w by summing over s in Equation 3, as seen in the table.

$f_{w,s}$	$s = 1$	2	3	4	5	6	$f_w = \sum_s f_{w,s}$
$w = 1$	1	0	0	0	0	0	1
2	1	1	0	0	0	0	2
3	2	2	1	0	0	0	5
4	5	5	3	1	0	0	14
5	14	14	9	4	1	0	42
6	42	42	28	14	5	1	132

The recurrence relation and values for $f_{w,s}$ coincide with that of a transpose of the so-called Catalan triangle, $C(n, k)$ in Equation 2.1 in [5]. Specifically, $f_{w,s} = C(w, w - s)$ for $1 \leq s \leq w$. In particular, the leftmost column in the table coincides with the Catalan numbers, as shown in [7].

A recursive procedure similar to tacking a slant onto a fountain in Equation 3 can be applied to three-dimensional mountains, as we see in the next section.

3 Mountains

Recall that a mountain is an arrangement of identical spheres into layers whose bottom layer forms a rectangular grid and whose higher layers follow the three-dimensional gravity rule. Also recall that $m_{n,w,\ell}$ is the number of mountains consisting of n spheres whose base has width w and length ℓ, and that $m_{w,\ell} = \sum_n m_{n,w,\ell}$ is the number of mountains with width w and length ℓ. In this section, we obtain a recurrence relation for $m_{n,w,\ell}$ which is used to derive a closed-form expression for $m_{w,\ell}$.

In a close analogy to fountains, we can formally consider mountains to be a collection of triples of positive integers in the following way. First, we imagine a mountain to be located in a three-dimensional Cartesian coordinate system with one corner of its base at the origin and the sides of the base aligned with the positive x and y-axes. Then each sphere in the mountain can be uniquely identified by the layer that it is in and its x and y-positions in this layer. We number the positions in each layer so that the x and y-positions of a sphere in any layer coincide with those of the farthest sphere from the origin among the four spheres supporting it in the layer below. Thus, in terms of integers, we can define a subset M of $\mathbb{N}^3$ to be a (n, w, ℓ) mountain if $|M| = n$ and in the bottom layer $(1, x, y) \in M$ for all $1 \leq x \leq \ell$ and $1 \leq y \leq w$ but $(1, x, y) \notin M$ for any other values, and for the gravity rule if $h > 1$ and $(h, x, y) \in M$, then so are $(h-1, x, y)$, $(h-1, x-1, y)$, $(h-1, x, y-1)$, and $(h-1, x-1, y-1)$. Two mountains are distinct if one has a sphere at some coordinate, but the other does not.

We first present some elementary properties of $m_{n,w,\ell}$ and $m_{w,\ell}$. By rotation, it is clear that $m_{w,\ell} = m_{\ell,w}$ for all w, ℓ. A mountain of width 1 could only consist of a single contiguous row of spheres, and so $m_{\ell,1,\ell} = m_{1,\ell} = 1$. As for mountains of width 2, the second horizontal layer has $\ell - 1$ places to insert spheres, so that $m_{2\ell+k,2,\ell} = \binom{\ell-1}{k}$ and $m_{2,\ell} = 2^{\ell-1}$.

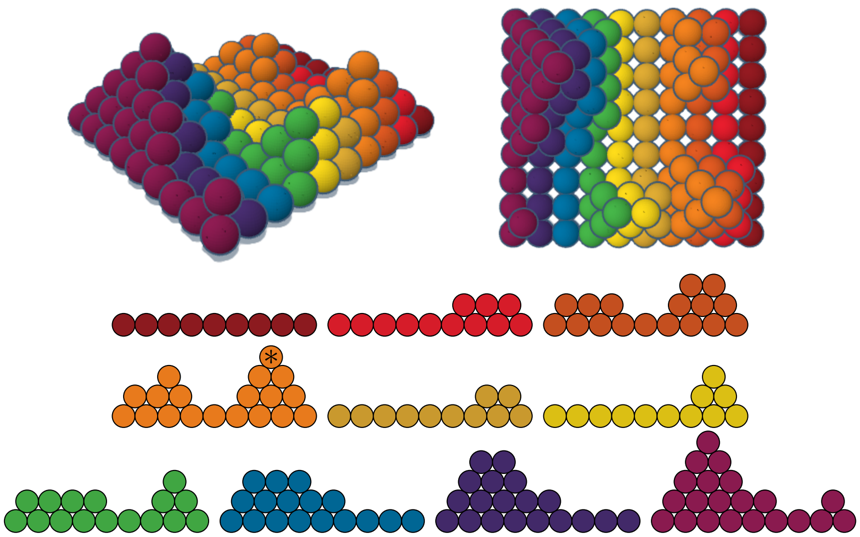


Figure 4: A $(153, 9, 10)$ mountain from oblique and top views and its decomposition into lateral faces. As an example of the coordinate system, the highest sphere in the light-orange lateral face, marked with a $*$, is located in layer 4 with x-position 4 and y-position 9.

Mountains can be decomposed into horizontal layers, as shown in Figure 3. However, just as fountains can be decomposed into a sequence of slants to enable an iterative method of building them, mountains can likewise be decomposed into lateral faces, as is shown in Figure 4. The k^{th} *lateral face* of a mountain is the set of spheres whose x-position is k, regardless of layer or y-position. What may be surprising is that each lateral face of a mountain actually forms a fountain when placed upright. This follows directly from the gravity rule, which ensures that any sphere with height greater than 1 lies atop four spheres in the layer below. Two of these spheres lie in the same lateral face as the original sphere, and are exactly those that must support it to satisfy the two-dimensional gravity rule for fountains. Similar arguments as those used for slants of fountains then enable any (n, w, ℓ) mountain to be expressed as an ℓ-tuple of fountains of width w.

We need a condition to test whether a sequence of fountains truly represents a mountain, analogous to Equation 2 for fountains. The crucial step is to determine when one fountain can "lean on" another. We say a fountain $F = \langle s_1, ..., s_w \rangle$ *leans on* a fountain $F' = \langle s'_1, ..., s'_w \rangle$, or conversely that F' *supports* F, and write $F' \succ F$ if when the coins in F are augmented to spheres with x-position ℓ and the coins in F' are augmented to spheres with x-position $\ell - 1$, then all the spheres in F satisfy the gravity rule. To characterize this leaning relation, consider two such fountains F and F'. The sphere at the top of the k^{th} slant in F has coordinates (s_k, ℓ, k) and by the gravity rule must be supported by the two spheres in F' with coordinates $(s_k - 1, \ell - 1, k - 1)$ and $(s_k - 1, \ell - 1, k)$. The slants s'_{k-1} and s'_k in F' must contain these two spheres that support (s_k, ℓ, k), and so $s'_{k-1} \geq s_k - 1$ and $s'_k \geq s_k - 1$. Therefore, the characterization for one fountain leaning on another is that $\langle s'_1, \ldots, s'_w \rangle \succ \langle s_1, \ldots, s_w \rangle$ if and only if both $s'_{k-1} \geq s_k - 1$ and $s'_k \geq s_k - 1$ for all $1 < k \leq w$. So, a sequence $F_1, \ldots, F_\ell$ of fountains of width w represents a mountain if and only if $F_1 = \mathbf{1}^w$, the sequence of w ones, and $F_1 \succ \cdots \succ F_\ell$. This characterization of mountains is a direct generalization of Equation 2 for fountains.

We may now apply the methods of Section 2 to iteratively build large mountains from small ones. An $(n', w, \ell - 1)$ mountain may be extended to an (n, w, ℓ) mountain with $n > n'$ by tacking on a fountain of $n - n'$ spheres and width w that can lean on the initial mountain's final lateral face. If we define $m_{n,w,\ell,F}$ to be the number of (n, w, ℓ) mountains whose final lateral face is the fountain F, then $m_{n,w,\ell} = \sum_F m_{n,w,\ell,F}$. Leaning a face F onto all mountains whose final lateral faces F' can support it gives the following recurrence relation:

$$\begin{aligned} m_{n,w,1,F} &= \begin{cases} 1 & \text{if } F = \mathbf{1}^w \text{ and } n = w \\ 0 & \text{otherwise.} \end{cases} \\ m_{n,w,\ell,F} &= \sum_{F' \succ F} m_{n-|F|,w,\ell-1,F'} \text{ for } \ell > 1. \end{aligned} \tag{4}$$

Equation 4 can be considered the mountain analog of Equation 3 for fountains and is the main tool for enumerating mountains. Using Maple to sum up Equation 4 over all w, ℓ and F's yields the numbers of mountains m_n for various numbers of spheres $n \leq 20$ as 1, 2, 2, 3, 3, 4, 6, 6, 9, 14, 18, 22, 33, 57, 76, 91, 139, 236, 348, 470, and, for example, $m_{63} = 29\,964\,295\,890$.

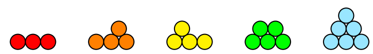
Equation 4 can also be used to enumerate mountains by the size of their bases. Summing over the number of spheres contained in each mountain, we have

$$\begin{aligned} m_{w,1,F} &= \begin{cases} 1 & \text{if } F = \mathbf{1}^w \\ 0 & \text{otherwise.} \end{cases} \\ m_{w,\ell,F} &= \sum_{F' \succ F} m_{w,\ell-1,F'} \text{ for } \ell > 1. \end{aligned} \tag{5}$$

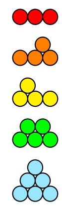
Note that Equation 5 is linear in the $m_{w,\ell,F}$'s (with coefficients of 1's or 0's) and so can be rewritten in terms of a matrix. We define the *leaning matrix* L_w of order w, which represents the relation that a fountain F of width w can lean on another fountain F' of width w, as follows:

$$L_w(F, F') = \begin{cases} 1 & \text{if } F' \succ F \\ 0 & \text{otherwise.} \end{cases} \tag{6}$$

So, L_w is a matrix of 0's and 1's whose rows and columns are indexed by fountains of width w in some specified order, say dictionary order on the numbers of spheres on the slants. The size of L_w is $C_w \times C_w$, where C_w is the w^{th} Catalan number. For instance, for $w = 3$



$$L_3 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$



By adding a zero term to the sum in Equation 5 for each F' that does not support F, we can express $m_{w,\ell,F}$ as a sum over *all* fountains using the leaning matrix

$$m_{w,\ell,F} = \sum_{F'} L_w(F, F') m_{w,\ell-1,F'} \quad . \tag{7}$$

To record the mountain counts themselves, we use column vectors. We define $\mathbf{m}_{w,\ell}$ to be the column vector of all $m_{w,\ell,F}$'s for the C_w many fountains of width w. We order the entries in the column vector with the dictionary order on the fountains' slant representations. In this ordering, $\mathbf{m}_{w,1}$ has a first entry of 1, followed by 0's. Each instance of Equation 7 may be recast as a dot product of the matrix L_w with the $\mathbf{m}_{w,\ell-1}$ vector, i.e., $\mathbf{m}_{w,\ell} = L_w \cdot \mathbf{m}_{w,\ell-1}$. After iterating this multiplication $\ell - 1$ times, we have $\mathbf{m}_{w,\ell} = L_w^{\ell-1} \cdot \mathbf{m}_{w,1}$. By summing over all lateral faces, we finally arrive a closed-form expression for the number of mountains with a specified base $m_{w,\ell}$, namely that $m_{w,\ell}$ is the sum of the entries in the first column of the $\ell - 1^{\text{th}}$ power of the leaning matrix L_w, i.e.,

$$m_{w,\ell} = \mathbf{1}^{C_w} \cdot L_w^{\ell-1} \cdot \mathbf{m}_{w,1} \quad . \tag{8}$$

For instance with $w = 3$, $m_{3,\ell} = [1, 1, 1, 1, 1] \cdot L_3^{\ell-1} \cdot [1, 0, 0, 0, 0]^t$. Equation 8 can be used to compute the number of mountains whose bases are small. The following table enumerates $m_{w,\ell}$ for $w, \ell \leq 6$.

$m_{w,\ell}$	$\ell = 1$	2	3	4	5	6
$w = 1$	1	1	1	1	1	1
2	1	2	4	8	16	32
3	1	4	17	73	314	1351
4	1	8	73	690	6583	62962
5	1	16	314	6583	141120	3048513
6	1	32	1351	62962	3048513	149892010

We note that the sequence $\langle m_{3,\ell} \rangle$ in the table has been shown in [2] to have the following ordinary generating function

$$A_3(x) = \frac{x(1-x)}{1-5x+3x^2}, \tag{9}$$

whose coefficients happen to coincide with OEIS sequence A018902 [9]. However, as of this writing, sequences $\langle m_{w,\ell} : \ell \geq 1 \rangle$ for $w \geq 4$ do not appear in OEIS.

4 Conclusion

While Equation 4 provides a recurrence relation to enumerate general fountains and Equation 8 provides an explicit formula for counting mountains based on the dimensions of their bases, these methods are hardly efficient. The leaning matrix that encodes which faces of width w can lean on which others is an $C_w \times C_w$ matrix, where C_w is the w^{th} Catalan number, approximately $4^w/w^{3/2}\sqrt{\pi}$. So, the size of the leaning matrix is exponential in the length of the mountain's base. Equation 8 also requires a power of the leaning matrix, but the power is small compared with the matrix's size. Therefore, it is unlikely that standard methods of accelerating computation of matrix powers, notably repeated squaring or diagonalization, would improve efficiency. One possible source of improvement would be to reduce the comparisons from *every pair* of faces by creating a small number of classes of faces to compare. For fountains, it is easy given a final slant s to partition slants as leaning on s and not leaning on s, but no such simple computation is known for lateral faces of mountains.

A likely next step in the analysis of mountains would be a computation of the generating functions of $\langle m_{n,w,\ell,F} \rangle$, $\langle m_{n,w,\ell} \rangle$, $\langle m_{w,\ell} \rangle$, and $\langle m_n \rangle$. Even though Equation 4 is linear, there are several reasons why the generating function for $\langle m_{m,w,\ell,F} \rangle$ is likely nontrivial. First, $m_{n,w,\ell,F}$ is indexed by a fountain, not just by integers. Second, the generating functions for mountains must generalize in some sense the generating functions for fountains, which, as seen in Equation 1 for Ramanujan's continued fraction, are already non-trivial. The problem of computing the generating functions for general classes of mountains is open.

We conclude with a related problem suggested by Serban Raianu. Instead of arranging spheres in the base of a mountain in a rectangle as part of a square lattice, the spheres can also be arranged in a parallelogram as part of a hexagonal lattice. In this case the gravity rule becomes simpler—each sphere above the base must sit on just the *three* spheres immediately below it. This arrangement gives the densest possible packing of spheres. With the spheres arranges in each layer as a hexagonal lattice, how many mountains are there with a $w \times \ell$ parallelogram base?

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