Polynomial Generalizations of Knot Colorings

R. He, A. Ho, D. Kalir, J. Miller, and M. Zevenbergen

Abstract - In the field of knot theory, knot invariants are properties preserved across all embeddings and projections of the same knot. Fox n-coloring is a classical knot invariant which associates to each knot projection a system of linear equations. We generalize Fox's n-coloring by using two, not necessarily distinct, polynomials over a field $\mathbb{F}$, which we say form a $(g, f)_{\mathbb{F}}$ coloring. We introduce a sufficient condition, called *strong*, for a pair of polynomials to form a $(g, f)_{\mathbb{F}}$ coloring. We confirm a family of pairs of linear polynomials each of which form a $(g, f)_{\mathbb{F}}$ coloring. We prove that there are no strong pairs containing an irreducible quadratic polynomial over a field $\mathbb{F}$ not of characteristic two. Furthermore, we find that the cubic $2x^3 - y^3 - z^3$ forms a $(g, f)_{\mathbb{F}_{3^n}}$ coloring and produce a method to find similar polynomials with unbounded degree.

Keywords : knot theory; knot invariants; Reidemeister moves

Mathematics Subject Classification (2020) **:** 57K10

1 Introduction

A knot is a simple closed curve in $\mathbb{R}^3$. Two knots K_1 and K_2 are considered equivalent if there exists an orientation preserving homeomorphism $p : \mathbb{R}^3 \to \mathbb{R}^3$ such that $p(K_1) = K_2$. Intuitively, this means that two knots are equivalent if one can manipulate one knot through $\mathbb{R}^3$, without intersecting itself, onto the other knot. We will focus on oriented knots, which are defined by specifying the direction a given knot can be traversed (signified by arrows in the knot projection).

One of the main problems in knot theory is determining if two knots are equivalent. One strategy to show that a pair of knots are not equivalent is to use *invariants*, or properties that are the same for every embedding of equivalent knots. One tool that many invariants use is the projection of a knot into the xy-plane. At any crossing, we can draw our projection so that we skip a neighborhood about the point with the lesser z coordinate, creating the illusion that one strand is passing over another. Formal projections as such, together with these gaps, are called *knot projections*, as depicted in Figure 1.

These gaps create *strands* in the projection, which are the connected components of the knot projection. At any crossing, we then have an overstrand, relative to the point with the greater z coordinate, and two understrands, local to the point with the lesser z coordinate.

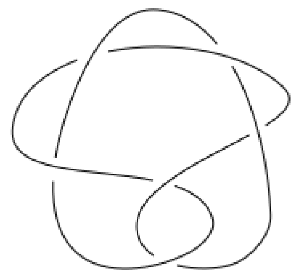

Figure 1: An example of a knot projection of the 6_1 knot.

1.1 Reidemeister Moves

One classical approach to determining if two knots are the same is using Reidemeister moves for knot projections [2]. Reidemeister moves are a type of modification to a knot projection, producing a distinct projection of the same knot. There are three classes of Reidemeister moves. There are variations within the three types of Reidemeister moves, but representatives of each of the three types are shown in Figure 2. We'll describe these in an informal way, which will suffice for our purposes. The first move has the effect of removing (or adding) a twist that occurs in a single knot strand. The second move has the effect of removing (or adding) a pair of crossings by allowing one strand to pass over/under another. The third Reidemeister move preserves the number of crossings, allowing one strand to pass over or under a crossing.

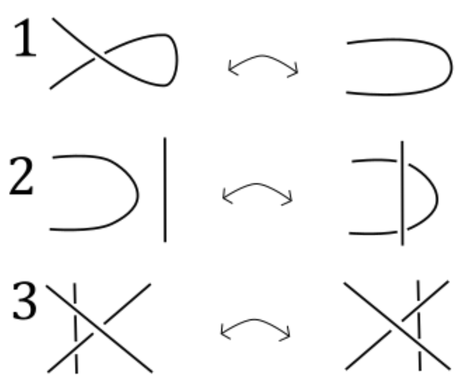

Figure 2: The three Reidemeister moves.

Reidemeister's main theorem regarding these moves is that if two knots K_1 and K_2 are equivalent if and only if one can use planar isotopies and a finite sequence of Reidemeister moves to obtain a projection of K_1 from a projection of K_2.

Reidemeister's theorem can be extended to oriented knots. However, there are quite a large number of these oriented Reidemeister moves. As discussed by Polyak [3], there are 4 oriented Reidemeister 1 moves, 4 oriented Reidemeister 2 moves, and 8 oriented

Reidemeister 3 moves. Polyak [3] shows, however, that the set of oriented Reidemeister moves $\Omega 1a, \Omega 1b, \Omega 2a, \Omega 3a$, shown in Figure 3, is a minimal generating set of oriented Reidemeister moves. That is to say, two oriented knots are equivalent if and only if one can use planar isotopies (continuous deformations of the plane of projection) and a finite sequence of $\Omega 1a, \Omega 1b, \Omega 2a, \Omega 3a$ moves to obtain a projection of K_1 from a projection of K_2, and there is no smaller set of oriented Reidemeister moves with this property.

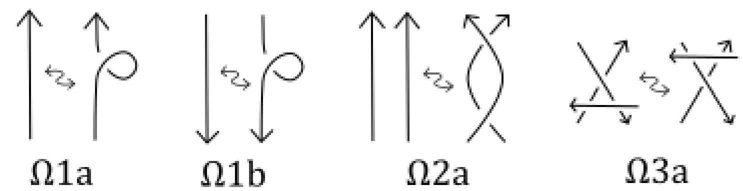

Figure 3: A minimal generating set of oriented Reidemeister moves.

1.2 Tricolorability and Generalizations

One classic knot invariant is *tricolorability.* We say a knot is tricolorable if one can assign to each strand of the knot projection one of three colors such that at each crossing all strands are either the same color, or all different (with at least two colors being used). One can show that this is equivalent to assigning each strand a 0, 1, or 2 and checking if at each crossing the equation

$$2x - y - z \equiv 0 \pmod 3$$

holds, where x is the label on the overstrand and y and z are the labels on the two understrands. Tricolorability can be generalized to picking labels from the set of integers $\{0, 1, ..., n-1\}$ and considering the equation $2x - y - z \equiv 0 \pmod n$ at each crossing. Such a labelling is called a *Fox n-coloring* [1]. If such a solution exists for a given knot projection, we say that the knot is n-colorable. It is a routine check that n-colorability is a knot invariant for all $n \geq 3$.

While the Fox n-coloring can differentiate some knots, there are many that it cannot. We examine alternative labeling methods than those which arise from Fox's work. One property of the Fox n-coloring is that the two understrands at a given crossing are not distinguished. If one adds an orientation to the knot, then there is a distinction between incoming and outgoing understrands at each crossing. Using this orientation, we can also distinguish between two types of crossings, which we will refer to as right-handed and left-handed crossings. Figure 4 depicts a right-hand crossing and a left-hand crossing.

Instead of using the same labeling equation at both of these crossings, we can instead consider using two different equations. This leads to the following definition:

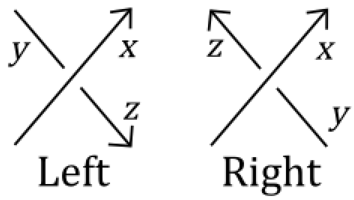

Figure 4: The two types of crossings possible in a knot projection.

Definition 1 *For polynomials $f, g \in \mathbb{Z}_n[x, y, z]$, a projection of an oriented knot is* $\boldsymbol{(g, f)_n}$ ***colorable*** *if each strand in a knot projection can be labeled with an integer from* 0 *to* $n-1$ *which satisfies the rules:*

1) *At least two different integers are used in the overall knot.*

2) $f(x, y, z) \equiv 0 \pmod{n}$ *at right-handed crossings and* $g(x, y, z) \equiv 0 \pmod{n}$ *at left-handed crossings, where x is the label of the overstrand, y is the label of the incoming understrand, and z is the label of the outgoing understrand (as shown in Figure 4).*

We call such a coloring a $\boldsymbol{(g, f)_n}$ ***coloring****.*

We can further generalize this definition by considering different rings from which to take labels. For the Fox n-coloring, labels were taken to be elements of $\mathbb{Z}_n$. We now consider taking polynomials over other rings, most of which we will choose to be fields.

Definition 2 *Let R be a ring with $|R| \geq 2$ and let $f, g \in R[x, y, z]$ be polynomials with coefficients in R. Then a projection of an oriented knot is* $\boldsymbol{(g, f)_R}$ ***colorable*** *if each strand in the knot projection can be labeled with an element of R such that:*

1) *At least two distinct elements of R are used in the coloring.*

2) $f(x, y, z) = 0$ *at right-handed crossings and* $g(x, y, z) = 0$ *at left-handed crossings, where x is the label of the overstrand, y is the label of the incoming understrand, and z is the label of the outgoing understrand.*

We call such a coloring a $\boldsymbol{(g, f)_R}$ ***coloring****.*

In Section 2, we derive a set of algebraic properties on pairs of polynomials from the Reidemeister moves and define a corresponding family of polynomial pairs called strong polynomials. We conclude this section with Theorem 9 by showing that colorability by strong pairs of polynomials is a knot invariant. In Section 3 we classify strong pairs of linear polynomials in Theorem 10. In Section 4, we show that there are no strong pairs of irreducible polynomials where either polynomial is quadratic. Finally, in Section 5 we describe a method of obtaining a pair of strong polynomials from another, forming an equivalent invariant, and use this to give strong pairs of polynomials with arbitrarily high degree.

2 Consequences of the Reidemeister Moves

We now consider a set of sufficient conditions for a $(g, f)_{\mathbb{F}}$ coloring to be invariant under the Reidemeister moves. These will then be sufficient conditions for the pair $(g, f)_{\mathbb{F}}$ to be a knot invariant. These conditions manifest as algebraic statements about the polynomials f and g, as well as their roots.

Definition 3 *Let $\mathbb{F}$ be a field and $f, g \in \mathbb{F}[x, y, z]$ be polynomials over $\mathbb{F}$. Then we define the following properties for the pair (f, g):*

- ***Property** $\Omega 1a_1$: $\forall \alpha \in \mathbb{F}$, $f(\alpha, \alpha, \alpha) = 0$*

- ***Property** $\Omega 1a_2$: $\forall \alpha, \beta \in \mathbb{F}$, $f(\alpha, \alpha, \beta) = 0$ implies $\alpha = \beta$*

- ***Property** $\Omega 1b_2$:[1] $\forall \alpha, \beta \in \mathbb{F}$, $f(\alpha, \beta, \alpha) = 0$ implies $\alpha = \beta$*

- ***Property** $\Omega 2a_1$: $\forall \alpha, \beta \in \mathbb{F}$, $\exists \gamma \in \mathbb{F}$ such that $f(\alpha, \beta, \gamma) = g(\alpha, \gamma, \beta) = 0$*

- ***Property** $\Omega 2a_2$: $\forall \alpha, \beta, \gamma, \delta \in \mathbb{F}$, $f(\alpha, \gamma, \delta) = g(\alpha, \delta, \beta) = 0$ implies $\beta = \gamma$*

- ***Property** $\Omega 3a$: $\forall \alpha, \beta, \gamma, \epsilon, \mu \in \mathbb{F}$ such that $f(\alpha, \mu, \beta) = 0$, $\exists \delta \in \mathbb{F}$ such that $f(\mu, \delta, \epsilon) = g(\alpha, \gamma, \delta) = 0$ if and only if $\exists \tau$ such that $f(\beta, \gamma, \tau) = g(\alpha, \tau, \epsilon) = 0$.*

*We say the pair (f, g) satisfies a certain property if that property is true for the pair (f, g). We call a pair of polynomials (f, g) **strong** if the pair (f, g) satisfies the above six properties.*

Note that these properties do not appear to be symmetric in f and g. We now show some additional properties that a strong pair of polynomials (f, g) has, and show that there is more symmetry between the implied restrictions on f and g than might be initially apparent.

Lemma 4 *Let $\mathbb{F}$ be a field with $f, g \in \mathbb{F}[x, y, z]$ such that (f, g) is strong. Then,*

a. (g, f) satisfies Property $\Omega 1a_1$

b. (g, f) satisfies Property $\Omega 1a_2$

c. (g, f) satisfies Property $\Omega 1b_2$

d. (g, f) satisfies Property $\Omega 2a_1$

[1] Note that what might be natural to call Property $\Omega 1b_1$ is the same as Property $\Omega 1a_1$.

e. (g, f) *satisfies Property* $\Omega 2a_2$

Proof. Note that we will prove these out of order.

(d.) Let $\alpha, \beta \in \mathbb{F}$. Since (f, g) satisfies $\Omega 1a_1$, we know $f(\alpha, \alpha, \alpha) = 0$. Utilizing (f, g) satisfying Property $\Omega 2a_1$, $\exists \tau \in \mathbb{F}$ such that $f(\alpha, \beta, \tau) = g(\alpha, \tau, \beta) = 0$. Then, $\Omega 3a$ tells us that $\exists \gamma \in \mathbb{F}$ such that $g(\alpha, \beta, \gamma) = f(\alpha, \gamma, \beta)$. This exactly tells us that (g, f) satisfies $\Omega 2a_1$.

(e.) Suppose we have $\alpha, \beta, \gamma, \delta \in \mathbb{F}$ such that $f(\alpha, \delta, \beta) = 0$ and $g(\alpha, \gamma, \delta) = 0$. Since (f, g) satisfies $\Omega 1a_1$, $f(\alpha, \alpha, \alpha) = 0$. Then, since (f, g) satisfies $\Omega 3a$, we know $\exists \tau \in \mathbb{F}$ such that $f(\alpha, \gamma, \tau) = g(\alpha, \tau, \beta) = 0$. Then, however, since (f, g) satisfies $\Omega 2a_2$, we must have $\beta = \gamma$. Thus, (g, f) satisfies $\Omega 2a_2$.

(c.) Suppose we have $\alpha, \beta \in \mathbb{F}$ such that $g(\alpha, \beta, \alpha) = 0$. Since (f, g) satisfies $\Omega 1a_1$, we have $f(\alpha, \alpha, \alpha) = 0$. Since part (*e.*) tells us that (g, f) satisfies $\Omega 2a_2$, we can then conclude $\alpha = \beta$. Thus, (g, f) satisfies $\Omega 1b_2$.

(a). Let $\alpha \in \mathbb{F}$. Then, since (f, g) satisfies $\Omega 2a_1$, we can say $\exists \gamma \in \mathbb{F}$ such that $f(\alpha, \alpha, \gamma) = g(\alpha, \gamma, \alpha) = 0$. Then, since (f, g) satisfies $\Omega 1a_2$, we must have $\alpha = \gamma$. Thus, $g(\alpha, \alpha, \alpha) = 0$, so (g, f) satisfies $\Omega 1a_1$.

(b.) Suppose we have $\alpha, \beta \in \mathbb{F}$ such that $g(\alpha, \alpha, \beta) = 0$. Since (f, g) satisfies $\Omega 1a_1$, we know $f(\alpha, \alpha, \alpha) = 0$. Then, since (f, g) satisfies $\Omega 2a_2$, we must have $\alpha = \beta$. Thus, (g, f) satisfies $\Omega 1a_2$. $\square$

We will now show that the pair (f, g) being strong is sufficient for $(g, f)_{\mathbb{F}}$ colorability to be a knot invariant.

Lemma 5 *Let* $\mathbb{F}$ *be a field with* $f, g \in \mathbb{F}[x, y, z]$. *If* (f, g) *satisfies Properties* $\Omega 1a_1$ *and* $\Omega 1a_2$*, then* $(g, f)_{\mathbb{F}}$ *colorability is invariant under Reidemeister move* $\Omega 1a$.

Proof. Suppose (f, g) satisfies Properties $\Omega 1a_1$ and $\Omega 1a_2$, and let H be a projection of an oriented knot which is $(g, f)_{\mathbb{F}}$ colorable. Consider any strand in a knot projection H, as depicted in Figure 5(a). Then, we consider a new knot projection H' which is the result of performing Reidemeister move $\Omega 1a$ to H to add a crossing to the strand. Because we assumed that H is colorable, there must exist a $(g, f)_{\mathbb{F}}$ coloring of all the strands in H. Let α be the label on the strand. Now, consider a coloring of H' given by coloring every strand in H' with the analogous label from H, with the exception that the two strands depicted on the right in Figure 5(a) are labeled α. Note that the expression associated to the crossing in H' is $f(\alpha, \alpha, \alpha)$, and since (f, g) satisfies Property $\Omega 1a_1$, the relation $f(\alpha, \alpha, \alpha) = 0$ holds for all α. Furthermore, we then see that because the coloring on H is valid, every other crossing in H' also satisfies the requisite relation, since each other crossing in H' has exactly the same labels as the analogous crossing in H. Additionally, we also note that at least two distinct labels are used, because the same number of distinct labels are used in H and H'. Thus, the coloring on H' is a valid $(g, f)_{\mathbb{F}}$ coloring, showing H' is $(g, f)_{\mathbb{F}}$ colorable. Now, to address the other direction of the $\Omega 1a$ Reidemeister move, let K be a projection of an oriented knot which is $(g, f)_{\mathbb{F}}$ colorable. Suppose K has a crossing as depicted in Figure 5(b). Then we can consider a new knot projection K' which is the result of performing the $\Omega 1a$ Reidemeister move

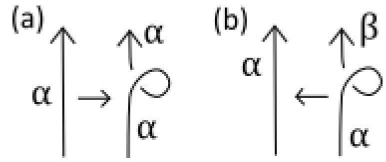

Figure 5: The two directions of the $\Omega 1a$ Reidemeister move.

to K at the crossing. Because we assumed that K is colorable, there must exist a $(g, f)_{\mathbb{F}}$ coloring of all strands in K. Considering such a coloring, let α and β be the labels on the corresponding strands on the right diagram of Figure 5(b). The associated relation with this crossing is $f(\alpha, \beta, \alpha) = 0$, as the crossing is right-handed. Since (g, f) satisfies Property $\Omega 1a_2$, we must have $\alpha = \beta$. Thus, since $\alpha = \beta$, the two strands depicted in the right diagram of 5(b) must have the same label. Hence, we can construct a $(g, f)_{\mathbb{F}}$ coloring for K' by coloring every strand in K' with the analogous label from K, with the exception of coloring the single strand depicted on the left in 5(a) with α. Notably, this coloring for K' is in fact a $(g, f)_{\mathbb{F}}$ coloring, as the strands in every crossing in K' have exactly the same labels as the analogous crossings in K. Thus, because the coloring on K is a valid coloring, each crossing satisfies $f(x, y, z) = 0$ or $g(x, y, z) = 0$ where x is the overstrand label, y is the incoming understrand label, and z is the outgoing understrand label. Additionally, we also note that at least two distinct labels are used, because the same number of distinct labels are used in K and K'. Thus, the coloring on K' is a valid $(g, f)_{\mathbb{F}}$ coloring, showing K' is $(g, f)_{\mathbb{F}}$ colorable.

Thus, $(g, f)_{\mathbb{F}}$ colorability is invariant under Reidemeister move $\Omega 1a$. □

Lemma 6 *Let $\mathbb{F}$ be a field with $f, g \in \mathbb{F}[x, y, z]$. If (f, g) satisfies Properties $\Omega 1a_1$ and $\Omega 1b_2$, then $(g, f)_{\mathbb{F}}$ colorability is invariant under Reidemeister move $\Omega 1b$.*

Proof. This lemma is proved nearly identically to the last lemma. Figure 6 provides the analogous local structures and labelings.

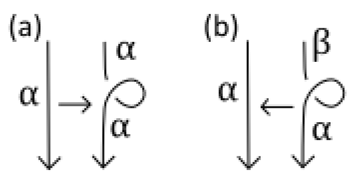

Figure 6: The two directions of the $\Omega 1b$ Reidemeister move.

□

Lemma 7 *Let $\mathbb{F}$ be a field with $f, g \in \mathbb{F}[x, y, z]$ as polynomials. If (f, g) is strong, then $(g, f)_{\mathbb{F}}$ colorability is invariant under Reidemeister move $\Omega 2b$.*

Proof. Suppose (f, g) is strong and consider Figure 7(a). Suppose H is a projection of an oriented knot which is $(g, f)_{\mathbb{F}}$ colorable. Suppose H has a pair of strands as depicted on the left in Figure 7(a). Then, we can consider a new knot projection H' which is identical to H except that the left-most strand is slid over the other strand, with the effect of adding two crossings. In other words, H' is the result of performing the $\Omega 2a$ Reidemeister move on H to add two crossings, as shown on the right of Figure 7(a). Because we assumed that H is colorable, there must exist a $(g, f)_{\mathbb{F}}$ coloring of all strands in H. Considering such a coloring, let α and β be the labels on the corresponding strands in the left diagram of Figure 7(a).

Now, note that since (f, g) satisfies Property $\Omega 2a_1$, there is some $\gamma \in \mathbb{F}$ such that $g(\alpha, \gamma, \beta) = 0$ and $f(\alpha, \beta, \gamma) = 0$. Consider a coloring on H' given by coloring every strand in H' with the analogous label from H, with the exception that the strands appearing in Figure 7(a) receive labels as shown. One can see that the expression associated with the top crossing in the diagram on the right is $g(\alpha, \gamma, \beta)$ and the expression associated with the bottom crossing is $f(\alpha, \beta, \gamma)$. Thus, since $g(\alpha, \gamma, \beta) = 0$ and $f(\alpha, \beta, \gamma) = 0$, as discussed above, and the relations associated with every other crossing in H' remain unchanged, one can note that the relation at every crossing in H' relative to a $(g, f)_{\mathbb{F}}$ coloring is valid. Furthermore, since the coloring on H' has at least as many distinct values as that on H, namely at least two, we have shown that H' has a valid $(g, f)_{\mathbb{F}}$ coloring.

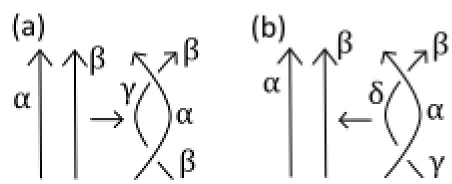

Figure 7: The two directions of the $\Omega 2a$ Reidemeister move.

Now, let K be a projection of an oriented knot which is $(g, f)_{\mathbb{F}}$ colorable and consider Figure 7(b). Suppose K has a pair of crossings as depicted on the right of Figure 7(b). Then, we can consider a new knot projection K' which is identical to K except that strands depicted in Figure 7(b) are slid apart with the effect of removing the two crossings. In other words, K' is the result of performing the $\Omega 2a$ Reidemeister move on K at the crossings in the right diagram of Figure 7(b), to produce the diagram to its left. Because we assumed that K is colorable, there must exist a $(g, f)_{\mathbb{F}}$ coloring of all strands in K. Considering such a coloring, let α, β, γ, δ be the labels on the corresponding strands on the right of Figure 7(a).

Note that since we have a valid coloring of K, there are at least two distinct labels associated to the strands in K. Note that either there are distinct labels in K outside Figure 7(b), or $\alpha \neq \beta$. To see this, suppose $\alpha = \beta$; then at the top-most crossing in the right in Figure 7(b) we have the equation $g(\alpha, \delta, \beta) = g(\alpha, \delta, \alpha) = 0$. Since (f, g) is strong, Lemma 4 tells us that (g, f) satisfies Property $\Omega 1b_2$, meaning $g(\alpha, \delta, \alpha) = 0$ implies $\alpha = \delta$.

We then get that the bottom-most crossing in the right in Figure 7(b) has the equation $f(\alpha,\gamma,\delta)=f(\alpha,\gamma,\alpha)=0$. By the same logic (f,g) satisfying $\Omega 1b_2$ implies $\gamma=\alpha$. Thus if $\alpha=\beta$, all labels in Figure 7(b) are equal, and so since we have a valid $(g,f)_{\mathbb{F}}$ coloring there must be some strand of K not shown in in Figure 7(b) that takes on a distinct label.

Now, consider the crossings in the right diagram of Figure 7(b). We have $g(\alpha,\delta,\beta)=0$ and $f(\alpha,\gamma,\delta)=0$. Since (f,g) is strong, the pair satisfies Property $\Omega 2a_2$ meaning these equations imply $\beta=\gamma$. Thus, we can form a coloring of K' by coloring the strands with their analogous labels in K, but coloring the rightmost strand in the left diagram of Figure 7(b) by $\beta=\gamma$. This is a $(g,f)_{\mathbb{F}}$ coloring of K' as either $\alpha\neq\beta$, in which case we have two distinct labels in K', or there are strands outside of Figure 7(b) that take on different labels, in which case these distinct labels are retained in K' outside 7(b). Additionally, each crossing will still be labeled by roots of either f or g as they were in K. Thus, K' is $(g,f)_{\mathbb{F}}$ colorable.

Thus, $(g,f)_{\mathbb{F}}$ colorability is invariant under Reidemeister move $\Omega 2a$. □

Lemma 8 *Let $\mathbb{F}$ be a field with $f,g\in\mathbb{F}[x,y,z]$. If (f,g) is strong, then $(g,f)_{\mathbb{F}}$ colorability is invariant under Reidemeister move $\Omega 3a$.*

Proof. Suppose (f,g) satisfies Property $\Omega 3a$, and suppose K is a projection of an oriented knot which is $(g,f)_{\mathbb{F}}$ colorable. Suppose K has 3 strands crossing as depicted on the left of Figure 8, and define K' to be the oriented knot diagram identical to K, with the exception that the top-most strand is translated above the central crossing, as depicted on the right in Figure 8. Since we assumed K was $(g,f)_{\mathbb{F}}$ colorable, there must exist a complete valid $(g,f)_{\mathbb{F}}$ coloring of K. Consider such a coloring, and let $\alpha,\beta,\gamma,\delta,\epsilon$ and μ be the labels of said coloring on the strands on the right in Figure 8.

Note that there must be two strands in K, neither of which are the strand labeled by δ, which have distinct labels under our valid coloring. To see this, suppose every label in K was equal aside from δ, so namely $\gamma=\alpha$. Then since (f,g) is strong, (g,f) satisfies Property $\Omega 1b_2$ by Lemma 4, and so considering the bottom-left crossing in the left diagram of Figure 8 we have the equation $g(\alpha,\delta,\gamma)=g(\alpha,\delta,\alpha)=0$ which implies that $\delta=\alpha$. This then implies that the coloring is not valid, as there are not 2 distinct labels in the coloring, and thus there must be two strands, neither of which are the one labeled by δ, which have different labels.

We now consider the 3 corresponding equations at each of the crossings in the left diagram of Figure 8: for the central one $f(\mu,\delta,\epsilon)=0$, for the bottom-left crossing $g(\alpha,\gamma,\delta)=0$, and for the bottom-right crossing $f(\alpha,\mu,\beta)=0$. Since (f,g) is strong, it satisfies Property $\Omega 3a$, and thus because δ exists there must exist some $\tau\in\mathbb{F}$ such that $f(\beta,\gamma,\tau)=0$ and $g(\alpha,\tau,\epsilon)=0$. We can then define a coloring of K' identical to the one used for K, with the exception that the middle segment originally labeled δ in K is labeled τ in K', as depicted on the right of Figure 8. This coloring is a valid $(g,f)_{\mathbb{F}}$ coloring of K', as each crossing not depicted in the right diagram of Figure 8 in K' is identical to the analogous crossing in K, and thus the labels at the crossing form a root of either f or g. For the new crossings in K', we defined τ specifically to satisfy both of the crossings $f(\beta,\gamma,\tau)=0$ and $g(\alpha,\tau,\epsilon)=0$, so each crossing in K' has labels satisfying either f or g. Additionally, this coloring is valid as the only label that is potentially not used from K is δ; however, we've shown there must be two strands with distinct labels that are not the δ strand in the left diagram of Figure 8. These labels and crossings would then be preserved in K', and thus K' has at least two distinct labels. Thus K' is $(g, f)_{\mathbb{F}}$ colorable.

The other direction is handled nearly identically. Thus, if (f, g) is strong, then $(g, f)_{\mathbb{F}}$ colorability is invariant under the $\Omega 3a$ Reidemeister move.

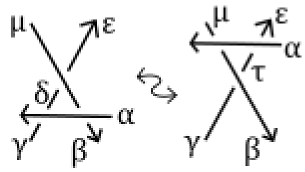

Figure 8: The two directions of the $\Omega 3a$ Reidemeister move.

□

We end this section by stating a theorem that serves to summarize the findings of this section.

Theorem 9 *If we have some field $\mathbb{F}$ and $f, g \in \mathbb{F}[x, y, z]$ such that the pair (f, g) is strong, then $(g, f)_{\mathbb{F}}$ colorability is an invariant for oriented knots.*

Proof. This theorem is a direct corollary of Lemmas 5,6,7,8 and Polyak's proof that invariance under the Reidemeister moves $\Omega 1a$, $\Omega 1b$, $\Omega 2a$, and $\Omega 3a$ implies being an oriented knot invariant [3]. □

3 Linear Case

In the following section, we provide a complete description of strong linear polynomial pairs, establishing exactly the form a pair of linear polynomials must have in order to be strong. We then conclude the only strong pair of linear polynomials (f, g) such that $f = g$ occurs when f is the Fox n-coloring polynomial $2x - y - z$.

Theorem 10 *Let $\mathbb{F}$ be a field and let $g, f \in \mathbb{F}[x, y, z]$ be linear polynomials. Then, (f, g) is strong if and only if $f(x, y, z) = ax + by - (a + b)z$ and $g(x, y, z) = cf(x, z, y)$ for some $a, b, c \in \mathbb{F}$ with $a + b, b, c \neq 0$.*

Proof. ($\Longrightarrow$) We start with arbitrary linear $g, f \in \mathbb{F}[x, y, z]$ such that $f(x, y, z) = ax + bx + cz + d$ and $g(x, y, z) = kx + ly + mz + n$. Suppose (f, g) is strong. Our proof will proceed by examining the restrictions that the properties place on the coefficients of f and g.

Since (f, g) is strong, (f, g) and (g, f) satisfy Property $\Omega 1 a_1$, so we can say that for any α,

$$\begin{aligned} 0 &= f(\alpha, \alpha, \alpha) = (a+b+c)\alpha + d \quad \text{and} \\ 0 &= g(\alpha, \alpha, \alpha) = (k+l+m)\alpha + n \end{aligned} \tag{1}$$

In particular, this must hold with $\alpha = 0$, so we must have $d = 0 = n$. Considering the equalities in (1) for $\alpha \neq 0$, note that this also tells us that $(a + b + c) = 0 = (k + l + m)$, since we clearly have no zero divisors. Thus, $c = -(a + b)$ and $m = -(k + l)$.

Now, note that since (f, g) and (g, f) satisfy Properties $\Omega 1 a_2$ and $\Omega 1 b_2$, we have that for all α, β, if any of the following four equations holds, then we must have that $\alpha = \beta$:

$$\begin{aligned} 0 &= f(\alpha, \alpha, \beta) = (a+b)\alpha + c\beta \\ 0 &= f(\alpha, \beta, \alpha) = (a+c)\alpha + b\beta \\ 0 &= g(\alpha, \alpha, \beta) = (k+l)\alpha + m\beta \\ 0 &= g(\alpha, \beta, \alpha) = (k+m)\alpha + l\beta. \end{aligned} \tag{2}$$

Recall that we determined above that $c = -(a + b)$ and $m = -(k + l)$, so we can simplify the above equations in (2) to say that if for any α, β any of the equations in (3) hold, then $\alpha = \beta$:

$$\begin{aligned} 0 &= c(\beta - \alpha) \\ 0 &= b(\beta - \alpha) \\ 0 &= m(\beta - \alpha) \\ 0 &= l(\beta - \alpha). \end{aligned} \tag{3}$$

In other words, each of the equations in (3) will not hold if $\beta - \alpha \neq 0$, which tells us that each of b, c, l, m must be non-zero.

Next, recall that since (g, f) satisfies Property $\Omega 2 a_2$, we must have that for any $\alpha, \beta, \gamma, \delta$, if $f(\alpha, \delta, \beta) = 0 = g(\alpha, \gamma, \delta)$, then we must have $\beta = \gamma$. Since $b \neq 0$ we can see that $f(\alpha, \delta, \beta) = 0$ if and only if $\delta = -(a\alpha + c\beta)/b$. Then, we can see that $f(\alpha, \delta, \beta) = 0 = g(\alpha, \gamma, \delta)$ if and only if

$$0 = k\alpha + l\gamma - \frac{m}{b}(a\alpha + c\beta) = (k - \frac{am}{b})\alpha + l\gamma - \frac{cm}{b}\beta. \tag{4}$$

Thus, if equality in (4) holds, then we must have $\beta = \gamma$. Since that conditional must hold for all α, β, γ, we must have that $k - \frac{am}{b} = 0$ and $l = \frac{cm}{b}$.

We can now say that

$$g(x, y, z) = \frac{am}{b}x + \frac{cm}{b}y + mz = \frac{m}{b}(ax + cy + bz)$$

where m/b is a unit in our field. We have $f(x, y, z) = ax + by + cz = ax + by - (a + b)z$, so this exactly tells us that $g(x, y, z) = (m/b)f(x, z, y)$. Noting that we have $b, c \neq 0$ and m/b a unit, we have completed this direction.

($\Longleftarrow$) Suppose we have $f, g \in \mathbb{F}[x, y, z]$ such that $f(x, y, z) = ax + by - (a + b)z$ and $g(x, y, z)$ is some unit multiple of $f(x, z, y)$ where $a, b \in \mathbb{F}$ with $a + b \neq 0 \neq b$.

Since multiplication by a unit does not affect zeros of a polynomial, we can assume without loss of generality that $g = f(x, z, y)$. So it will suffice to show that (f, g) satisfies Properties.

First, suppose we have α, β such that $f(\alpha, \alpha, \beta) = 0$. Then, $(a + b)(\alpha - \beta) = 0$, so since $a + b \neq 0$, we must have $\alpha = \beta$, demonstrating $\Omega 1a_2$. Furthermore, note that it is clear that (f, g) satisfies Property $\Omega 1a_1$.

Now, suppose we have $\alpha, \beta, \gamma, \delta$ such that $f(\alpha, \gamma, \delta) = 0 = g(\alpha, \delta, \beta)$. Then

$$a\alpha + b\gamma - (a + b)\delta = 0 = a\alpha - (a + b)\delta + b\beta,$$

so since $b \neq 0$, simplifying reveals that $\gamma = \delta$. Hence, (f, g) satisfies property $\Omega 2a_2$

Now, suppose we have arbitrary α, β. Then, noting that $a + b \neq 0$, letting

$$\gamma = \frac{a\alpha + b\beta}{a + b}$$

we can note that $f(\alpha, \beta, \gamma) = g(\alpha, \gamma, \beta) = 0$. Thus, (f, g) satisfies Property $\Omega 2a_1$.

It remains to show that (f, g) satisfies Property $\Omega 3a$. Suppose we have $\alpha, \beta, \gamma, \epsilon, \mu$ with $f(\alpha, \mu, \beta) = 0$. Now suppose there exists a δ such that $f(\mu, \delta, \epsilon) = g(\alpha, \gamma, \delta) = 0$. Then, substituting, we have $a\mu + b\delta - (a+b)\epsilon = a\alpha - (a+b)\gamma + b\delta$, so $a\mu - (a+b)\epsilon = a\alpha - (a+b)\gamma$. We also have $0 = f(\alpha, \mu, \beta) = a\alpha + b\mu - (a + b)\beta$, so since $b \neq 0$ we can solve for μ to get $\mu = ((a + b)\beta - a\alpha)/b$. Then we can combine the equations we have and say

$$a\alpha - (a + b)\gamma = a\frac{(a + b)\beta - a\alpha}{b} - (a + b)\epsilon.$$

Then, $ab\alpha - b(a+b)\gamma = a(a+b)\beta - a^2\alpha - b(a+b)\epsilon$, which we can simplify to $(a+b)(a\alpha+b\epsilon) = (a+b)(a\beta+b\gamma)$. Since $a+b \neq 0$, we can divide both sides by $a+b$ to get $a\beta+b\gamma = a\alpha+b\epsilon$. Now, let $\tau = \frac{a\beta+b\gamma}{a+b}$. Then we see $f(\beta, \gamma, \tau) = a\beta + b\gamma - (a + b)(a\beta + b\gamma)/(a + b) = 0$ and $g(\alpha, \tau, \epsilon) = a\alpha - (a + b)(a\alpha + b\epsilon)/(a + b) + b\epsilon = 0$. Note the converse direction has an analogous proof.

Thus, we conclude that (f, g) satisfies Property $\Omega 3a$, and we have indeed shown that (f, g) is strong. □

Corollary 11 *The only linear polynomials f for which (f, f) is strong are multiples of the Fox n-coloring polynomial, $2x - y - z$, or multiples of $y - z$.*

Proof. By the previous theorem, the only linear polynomials that are strong are of the form $f(x, y, z) = ax + by - (a + b)z$ and $g(x, y, z) = cf(x, z, y) = cax + -c(a + b)y + cbz$ where $a + b, b, c \neq 0$. If $f = g$, then we have $ax + by - (a + b)z = cax - c(a + b)y + cbz$ so $a = ca$, $b = -c(a + b)$, and $-(a + b) = cb$. Substituting the last two equations together yields that $b = c(cb) = c^2b$, and since $b \neq 0$ we can divide to get that $c^2 = 1$, meaning $c = \pm 1$. If $c = -1$, then $a = -a$ meaning $a = 0$, and thus $f(x, y, z) = by - bz = b(y - z)$. Now if $c = 1$, then $a = a$ and $b = -(a + b)$ meaning $a = -2b$. Thus, $f(x, y, z) = -2bx + by + bz = -b(2x - y - z)$. Thus, f is either a multiple of $2x - y - z$ or a multiple of $y - z$. □

4 Quadratic Case

Let $\mathbb{F}$ be a field and let $p \in \mathbb{F}[x, y, z]$. Then, p is said to be quadratic if p is of the form

$$p(x, y, z) = a_0 + a_1x + a_2y + a_3z + a_4x^2 + a_5y^2 + a_6z^2 + a_7xy + a_8xz + a_9yz,$$

with coefficients in $\mathbb{F}$. A quadratic polynomial p will be said to be irreducible if it cannot be written as the product of two non-constant polynomials. In this section, we prove that there are no strong pairs of polynomials where one of the polynomials is an irreducible quadratic.

The following is the main result of this section. In the proof, we bold certain relations to bring the reader's attention to them.

Theorem 12 *Let $\mathbb{F}$ be a field with characteristic not equal to 2. Then for any pair of polynomials $f, g \in \mathbb{F}[x, y, z]$, if (f, g) is strong, then neither f nor g are irreducible quadratics.*

Proof. This proof will not utilize $\Omega 3a$, and so by Lemma 4 we note every property used in the proof is satisfied exactly by both f and g. Hence, it suffices to demonstrate that f must factor, and we need not consider the form of g separately.

Let $f(x, y, z) = a_0 + a_1x + a_2y + a_3z + a_4x^2 + a_5y^2 + a_6z^2 + a_7xy + a_8xz + a_9yz$ be a quadratic polynomial in $\mathbb{F}[x, y, z]$, and suppose that (f, g) is strong. Then (f, g) satisfies Property $\Omega 1a_1$, so $\forall \alpha \in \mathbb{F}$, $f(\alpha, \alpha, \alpha) = 0$. We can plug this in to get that

$$f(\alpha, \alpha, \alpha) = a_0 + (a_1 + a_2 + a_3)\alpha + (a_4 + a_5 + a_6 + a_7 + a_8 + a_9)\alpha^2 = 0.$$

Since this holds for any $\alpha \in \mathbb{F}$, consider $\alpha = 0$. This gives us that $\mathbf{a_0 = 0}$, meaning we can now factor $f(\alpha, \alpha, \alpha)$ to the form

$$f(\alpha, \alpha, \alpha) = \alpha[(a_1 + a_2 + a_3) + (a_4 + a_5 + a_6 + a_7 + a_8 + a_9)\alpha] = 0.$$

For any $\alpha \neq 0$, we can simplify this expression by dividing by α to get

$$(a_1 + a_2 + a_3) + (a_4 + a_5 + a_6 + a_7 + a_8 + a_9)\alpha = 0.$$

The equality above must hold for all $\alpha \neq 0$; however, this is a linear expression in α, and thus will have only one solution if $a_4 + a_5 + a_6 + a_7 + a_8 + a_9 \neq 0$. Having only one solution would contradict Property $\Omega 1a_1$; thus, we must have that

$$a_4 + a_5 + a_6 + a_7 + a_8 + a_9 = 0.$$

Furthermore, for $\alpha \neq 0$, $f(\alpha, \alpha, \alpha)$ will not be equal to zero unless $a_1 + a_2 + a_3 = 0$. Thus as a result of Property $\Omega 1a_1$, we have that the following two equations hold on the coefficients of f, given $a_0 = 0$:

$$a_1 + a_2 + a_3 = 0 \tag{5}$$

$$a_4 + a_5 + a_6 + a_7 + a_8 + a_9 = 0. \tag{6}$$

Hence, in order to satisfy Property $\Omega 1 a_1$, $f(x,y,z) = a_1x + a_2y + a_3z + a_4x^2 + a_5y^2 + a_6z^2 + a_7xy + a_8xz + a_9yz$ where (5) and (6) hold.

From here, there are 4 different cases, each with a slightly different approach. Either (1) both $a_5, a_6 = 0$, (2) $a_5 \neq 0$ but $a_6 = 0$, (3) $a_5 = 0$ but $a_6 \neq 0$, or (4) both $a_5, a_6 \neq 0$. However, note that any of the properties $f(x,y,z)$ satisfies, $f(x,z,y)$ also satisfies as (f,g) and (g,f) satisfy the same conditions by Lemma 4. Hence, case 3 for $f(x,y,z)$ is equivalent to case 2 for $f(x,z,y)$, as the coefficients are determined by the monomial they scale. So showing that f factors in case 2 demonstrates that $f(x,z,y)$ also factors in its case 2, which is case 3 for $f(x,y,z)$. Hence, these cases are equivalent, and it suffices to show f factors in case 2. Thus, there are only 3 cases in total to consider.

Case 1: Suppose that $a_5, a_6 = 0$. Since f satisfies Property $\Omega 1 a_1$ it must be of the form $f(x,y,z) = a_1x + a_2y + a_3z + a_4x^2 + a_7xy + a_8xz + a_9yz$ where (5) holds and (6) can be restated plugging in the new values of a_5 and a_6 as:

$$a_4 + a_7 + a_8 + a_9 = 0. \tag{7}$$

Consider that f satisfies Property $\Omega 1 a_2$, meaning $\forall \alpha, \beta \in \mathbb{F}$ we have $f(\alpha, \alpha, \beta) = 0$ implies that $\alpha = \beta$. Note that such roots must exist, as by Property $\Omega 2 a_1$ we have that $\forall \alpha, \beta \in \mathbb{F}\ \exists \gamma \in \mathbb{F}$ such that $f(\alpha, \beta, \gamma) = 0$. Taking $\alpha = \beta$ gives that $\exists \gamma \in \mathbb{F}$ such that $f(\alpha, \alpha, \gamma) = 0$, which is a root of the desired form. So f cannot vacuously satisfy the property.
We can plug in the values into $f(\alpha, \alpha, \beta)$ to see that for any α, β:

$$f(\alpha, \alpha, \beta) = (a_1 + a_2)\alpha + a_3\beta + (a_4 + a_7)\alpha^2 + (a_8 + a_9)\alpha\beta = 0.$$

Rearranging (5) and (7) shows that $a_1 + a_2 = -a_3$ and $a_4 + a_7 = -(a_8 + a_9)$ respectively. Substituting these values in gives that

$$f(\alpha, \alpha, \beta) = -a_3\alpha + a_3\beta - (a_8 + a_9)\alpha^2 + (a_8 + a_9)\alpha\beta = (\beta - \alpha)(a_3 + (a_8 + a_9)\alpha) = 0.$$

Note that if $a_8 + a_9 \neq 0$, then there is a singular value $\alpha_0 \in \mathbb{F}$ such that $a_3 + (a_8 + a_9)\alpha_0 = 0$. Thus, we can take any β, particularly $\beta \neq \alpha_0$, and have that

$$f(\alpha_0, \alpha_0, \beta) = (\beta - \alpha_0)(a_3 + (a_8 + a_9)\alpha_0) = 0.$$

This violates Property $\Omega 1 a_2$, and so we must have $a_8 + a_9 = 0$ meaning $\mathbf{a_9 = -a_8}$. Note that this implies that (7) becomes $a_4 + a_7 = 0$, or $\mathbf{a_7 = -a_4}$. Furthermore, if $a_3 = 0$, then we can pick any value of α and β independently as $f(\alpha, \alpha, \beta) \equiv 0$, which also violates Property $\Omega 1 a_2$. Thus, $\mathbf{a_3 \neq 0}$, meaning in order for f to be quadratic and satisfy Properties $\Omega 1 a_1$ and $\Omega 1 a_2$, $f(x,y,z) = a_1x + a_2y + a_3z + a_4x^2 - a_4xy + a_8xz - a_8yz$ where $a_3 \neq 0$ and (5) holds.
Now consider that f also satisfies Property $\Omega 1 b_2$, so $\forall \alpha, \beta \in \mathbb{F}$ we have $f(\alpha, \beta, \alpha) = 0$ implies that $\alpha = \beta$. Since (g,f) satisfies Property $\Omega 2 a_1$ by assumption, by a similar argument as above there is a root of f of the form $(\alpha, \beta, \alpha) \in \mathbb{F}^3$, so f cannot vacuously satisfy this property as well. Evaluating $f(\alpha, \beta, \alpha)$ gives that

$$f(\alpha, \beta, \alpha) = (a_1 + a_3)\alpha + a_2\beta + (a_4 + a_8)\alpha^2 - (a_4 + a_8)\alpha\beta = 0.$$

Rearranging (5) gives that $a_1 + a_3 = -a_2$, so we can once again simplify the above expression into

$$f(\alpha,\beta,\alpha) = -a_2\alpha + a_2\beta + (a_4+a_8)\alpha^2 - (a_4+a_8)\alpha\beta = (\beta-\alpha)(a_2 - (a_4+a_8)\alpha) = 0.$$

This is exactly analogous to the previous condition, meaning that in order to satisfy Property $\Omega 1b_2$ we need that $a_4 + a_8 = 0$ or $\mathbf{a_8 = -a_4}$ and $\mathbf{a_2 \neq 0}$. Thus, for f to satisfy Properties $\Omega 1a_1$, $\Omega 1a_2$, and $\Omega 1b_1$ respectively, we need that $f(x,y,z) = a_1x + a_2y + a_3z + a_4x^2 - a_4xy - a_4xz + a_4yz$ where $a_2, a_3 \neq 0$ and (5) holds.
Now, consider that f satisfies Property $\Omega 2a_1$, i.e. $\forall \alpha,\beta \in \mathbb{F}\ \exists \gamma \in \mathbb{F}$ such that $f(\alpha,\beta,\gamma) = 0$. Substituting gives

$$\begin{aligned} f(\alpha,\beta,\gamma) &= a_1\alpha + a_2\beta + a_3\gamma + a_4\alpha^2 - a_4\alpha\beta - a_4\alpha\gamma + a_4\beta\gamma \\ &= (a_3 + a_4[\beta-\alpha])\gamma + a_1\alpha + a_2\beta + a_4\alpha^2 - a_4\alpha\beta \\ &= 0. \end{aligned}$$

We can move the γ term to the other side to see that for any $\alpha,\beta \in \mathbb{F}$ there is a $\gamma \in \mathbb{F}$ such that

$$(a_4[\alpha-\beta] - a_3)\gamma = a_1\alpha + a_2\beta + a_4\alpha^2 - a_4\alpha\beta.$$

Since this holds for any α and β, it must hold for $\beta = 0$, meaning for any $\alpha \in \mathbb{F}$ we have that

$$(a_4\alpha - a_3)\gamma = \alpha(a_1 + a_4\alpha)$$

holds for some $\gamma \in \mathbb{F}$. Suppose $a_4 \neq 0$; then there is a lone value of α, α_0 such that $a_4\alpha_0 - a_3 = 0$. In fact, $\alpha_0 = \frac{a_3}{a_4}$, which is nonzero. Using this to evaluate $a_1 + a_4\alpha$ gives that $a_1 + a_4\frac{a_3}{a_4} = a_1 + a_3 = -a_2$ is nonzero. Thus, the right-hand side equals $-\frac{a_2a_3}{a_4}$, i.e. we have $0\gamma = 0 = -\frac{a_2a_3}{a_4}$ which is impossible for any $\gamma \in \mathbb{F}$. Thus, for $\alpha = \frac{a_3}{a_4}$ and $\beta = 0$, there is no γ such that $f(\alpha,\beta,\gamma) = 0$, which violates Property $\Omega 2a_1$. Hence, $a_4 = 0$, but this means that f is of the form $a_1x + a_2y + a_3z$, which is a linear polynomial. Hence, f is no longer a quadratic polynomial, meaning there is no quadratic polynomial f with $a_5, a_6 = 0$ such that both (f,g) and (g,f) satisfy Properties $\Omega 1a_1$, $\Omega 1a_2$, $\Omega 1b_2$, and $\Omega 2a_1$ respectively.

Cases 2 & 3: Now suppose $a_5 \neq 0$ but $a_6 = 0$, so for f to satisfy Property $\Omega 1a_1$ we need that $f(x,y,z) = a_1x + a_2y + a_3z + a_4x^2 + a_5y^2 + a_7xy + a_8xz + a_9yz$ where (5) holds as stated but (6) becomes

$$a_4 + a_5 + a_7 + a_8 + a_9 = 0. \tag{8}$$

Since f satisfies Property $\Omega 1a_2$, $\forall \alpha,\beta \in \mathbb{F}$ we have that $f(\alpha,\alpha,\beta) = 0$ implies $\alpha = \beta$. Such a root must exist by Property $\Omega 2a_1$, so f cannot satisfy this conditional vacuously. Substituting gives that

$$f(\alpha,\alpha,\beta) = (a_1+a_2)\alpha + a_3\beta + (a_4+a_5+a_7)\alpha^2 + (a_8+a_9)\alpha\beta = 0.$$

Note that by rearranging (5) and (8) we see that $a_1 + a_2 = -a_3$ and $a_4 + a_5 + a_7 = -(a_8 + a_9)$, meaning we have that

$$f(\alpha,\alpha,\beta) = -a_3\alpha + a_3\beta - (a_8+a_9)\alpha^2 + (a_8+a_9)\alpha\beta = (\beta-\alpha)(a_3 + (a_8+a_9)\alpha) = 0.$$

This is the same as in Case 1, so we can conclude that $\mathbf{a_9} = -\mathbf{a_8}$ and $\mathbf{a_3} \neq \mathbf{0}$. This means that (8) now reads as

$$a_4 + a_5 + a_7 = 0 \tag{9}$$

So for f to satisfy Properties $\Omega 1a_1$, $\Omega 1a_2$, and $\Omega 2a_1$, f must be of the form $f(x, y, z) = a_1x + a_2y + a_3z + a_4x^2 + a_5y^2 + a_7xy + a_8xz - a_8yz$ where $a_3 \neq 0$, (5) holds, and (9) holds.

Now consider that f satisfies Property $\Omega 1b_2$, i.e. $\forall \alpha, \beta \in \mathbb{F}$ we have that $f(\alpha, \beta, \alpha) = 0$ implies $\alpha = \beta$. Like before, Property $\Omega 2a_1$ prevents f from satisfying this implication vacuously. We then have

$$f(\alpha, \beta, \alpha) = (a_1 + a_3)\alpha + a_2\beta + (a_4 + a_8)\alpha^2 + a_5\beta^2 + (a_7 - a_8)\alpha\beta = 0.$$

Since $a_5 \neq 0$, we can rearrange the equation into the form of a univariate polynomial in β with coefficients in $\mathbb{F}[\alpha]$ to get that

$$a_5\beta^2 + (a_2 + [a_7 - a_8]\alpha)\beta + (a_1 + a_3)\alpha + (a_4 + a_8)\alpha^2 = 0.$$

We can then use the quadratic formula as $\mathbb{F}$ is not of characteristic 2 to get that

$$\beta = \frac{([a_8 - a_7]\alpha - a_2) \pm \sqrt{([a_8 - a_7]\alpha - a_2)^2 - 4a_5\alpha(a_1 + a_3 + [a_4 + a_8]\alpha)}}{2a_5}.$$

Since we need that this implies $\alpha = \beta$, the discriminant must be 0, as if it wasn't then β would have two distinct values, one of which would not be α. This would then imply there is an $\alpha \neq \beta$ such that $f(\alpha, \beta, \alpha) = 0$, which violates Property $\Omega 1b_2$. Thus, the discriminant must be 0, giving $\beta = \frac{(a_8 - a_7)\alpha - a_2}{2a_5}$. We know that this equality must reduce to $\beta = \alpha$ for any α, so we get that

$$\frac{(a_8 - a_7)\alpha - a_2}{2a_5} = \alpha \Longrightarrow (a_8 - a_7)\alpha - a_2 = 2a_5\alpha$$

which is only possible if $a_8 - a_7 = 2a_5$ or $\mathbf{a_8} = \mathbf{2a_5} + \mathbf{a_7}$ and $\mathbf{a_2} = \mathbf{0}$. Note that $a_2 = 0$ means that (5) becomes $a_1 + a_3 = 0$ or $\mathbf{a_3} = -\mathbf{a_1}$. Using these new relations, we can simplify the discriminant to get that

$$\begin{aligned}([a_8 - a_7]\alpha - a_2)^2 - 4a_5\alpha(a_1 + a_3 + [a_4 + a_8]\alpha) &= (2a_5\alpha)^2 - 4a_5\alpha^2(a_4 + 2a_5 + a_7)\\ &= 4a_5\alpha^2(a_5 - a_4 - 2a_5 - a_7)\\ &= -4a_5\alpha^2(a_4 + a_5 + a_7).\end{aligned}$$

However, (9) states that $a_4 + a_5 + a_7 = 0$, so the discriminant is 0, as desired. Thus, for f to satisfy Properties $\Omega 1a_1$, $\Omega 1a_2$, $\Omega 1b_2$, and $\Omega 2a_1$ given $a_5 \neq 0$ and $a_6 = 0$, f must be of the form $f(x, y, z) = a_1x - a_1z + a_4x^2 + a_5y^2 + a_7xy + (2a_5 + a_7)xz - (2a_5 + a_7)yz$ where $a_1 \neq 0$ and (9) holds.

Now let's consider that f satisfies Property $\Omega 2a_1$, meaning $\forall \alpha, \beta \in \mathbb{F}\ \exists \gamma \in \mathbb{F}$ such that $f(\alpha, \beta, \gamma) = 0$. We then have

$$f(\alpha, \beta, \gamma) = a_1\alpha - a_1\gamma + a_4\alpha^2 + a_5\beta^2 + a_7\alpha\beta + (2a_5 + a_7)\alpha\gamma - (2a_5 + a_7)\beta\gamma = 0.$$

We can rearrange for γ to get that

$$([2a_5 + a_7][\alpha - \beta] - a_1)\gamma + a_1\alpha + a_4\alpha^2 + a_5\beta^2 + a_7\alpha\beta = 0$$
$$\Downarrow$$
$$(a_1 + [2a_5 + a_7][\beta - \alpha])\gamma = a_1\alpha + a_4\alpha^2 + a_5\beta^2 + a_7\alpha\beta.$$

Since this must hold for any α, we can take $\alpha = 0$ and get that for any $\beta \in \mathbb{F}$ there is some $\gamma \in \mathbb{F}$ such that

$$([2a_5 + a_7]\beta + a_1)\gamma = a_5\beta^2.$$

Supposing that $2a_5 + a_7 \neq 0$, we may find a specific value $\beta_0 \neq 0$ for β such that $(2a_5 + a_7)\beta_0 + a_1 = 0$. Taking this value of β gives that $a_5\beta_0^2 = 0\gamma = 0$. However, $a_5 \neq 0$, and since $\beta_0 \neq 0$, we have $\beta_0^2 \neq 0$. Thus, this equation is impossible, meaning $\forall\gamma \in \mathbb{F}$, $f(0, \beta_0, \gamma) \neq 0$. This violates Property $\Omega 2a_1$, and so $2a_5 + a_7 = 0$, or $\mathbf{a_7 = -2a_5}$. Using this we can evaluate (9) to get $a_4 + a_5 - 2a_5 = a_4 - a_5 = 0$, or $\mathbf{a_4 = a_5}$. This then simplifies $f(\alpha, \beta, \gamma)$ to saying

$$a_1\gamma = a_1\alpha + a_5\alpha^2 + a_5\beta^2 - 2a_5\alpha\beta.$$

Since $a_1 \neq 0$, such a γ will always exist for any $\alpha, \beta \in \mathbb{F}$, satisfying the Property. Thus, for f to satisfy the Properties given $a_5 \neq 0$, $a_6 = 0$, f must be of the form $f(x, y, z) = a_1x - a_1y + a_5x^2 + a_5y^2 - 2a_5xy$ where $a_1 \neq 0$. However, this can be factored into $f(x, y, z) = a_1(y - x) + a_5(y - x)^2 = (y - x)(a_5y - a_5x + a_1)$. Thus, f is not an irreducible quadratic. In conclusion, there is no irreducible quadratic f with coefficients $a_5 \neq 0$, $a_6 = 0$ satisfying the Properties. As discussed, this also implies there is no irreducible quadratic f satisfying the same properties if $a_5 = 0$ and $a_6 \neq 0$, proving Case 3 as well.

Case 4: Finally, suppose $a_5, a_6 \neq 0$, so for f to satisfy $\Omega 1a_1$, we need f to be of the form $f(x, y, z) = a_1x + a_2y + a_3z + a_4x^2 + a_5y^2 + a_6z^2 + a_7xy + a_8xz + a_9yz$ where (5) and (6) hold as stated. f satisfies Property $\Omega 1a_2$, meaning $\forall\alpha, \beta \in \mathbb{F}$ we have $f(\alpha, \alpha, \beta) = 0$ implies $\alpha = \beta$. This cannot hold, vacuously, by Property $\Omega 2a_1$. Substitution yields

$$f(\alpha, \alpha, \beta) = (a_1 + a_2)\alpha + a_3\beta + (a_4 + a_5 + a_7)\alpha^2 + a_6\beta^2 + (a_8 + a_9)\alpha\beta = 0.$$

We can simplify the above using (5) and (6) as $a_1 + a_2 = -a_3$ and $a_4 + a_5 + a_7 = -(a_6 + a_8 + a_9)$ yielding:

$$\begin{aligned} f(\alpha, \alpha, \beta) &= -a_3\alpha + a_3\beta - a_6\alpha^2 + a_6\beta^2 - (a_8 + a_9)\alpha^2 + (a_8 + a_9)\alpha\beta \\ &= (\beta - \alpha)(a_3 + a_6[\beta + \alpha] + [a_8 + a_9]\alpha) = 0. \end{aligned}$$

Note that since $a_3 + a_6(\beta + \alpha) + (a_8 + a_9)\alpha$ is linear in α and β, it will have zeroes in $\mathbb{F}$. We need these zeroes to occur only when $\alpha = \beta$, otherwise we violate Property $\Omega 1a_2$. Thus, we need that

$$a_3 + 2a_6\alpha + (a_8 + a_9)\alpha = a_3 + (2a_6 + a_8 + a_9)\alpha = 0$$

for any α. This only occurs if $\mathbf{a_3 = 0}$ and $2a_6 + a_8 + a_9 = 0$ or $\mathbf{a_9 = -(2a_6 + a_8)}$. The first bold equation implies that $\mathbf{a_2 = -a_1}$ by (5) and the second bold equation implies that

$$a_4 + a_5 + a_7 - a_6 = 0 \tag{10}$$

by (6). Note that these values of the coefficients make

$$a_3 + a_6(\beta+\alpha) + (a_8+a_9)\alpha = a_6(\beta+\alpha) - 2a_6\alpha = a_6(\beta-\alpha)$$

and so

$$f(\alpha,\alpha,\beta) = a_6(\beta-\alpha)^2$$

which is only ever 0 when $\alpha=\beta$, as desired. Thus, for f to satisfy Property $\Omega 1a_2$ and Property $\Omega 1a_1$ given $a_5, a_6 \neq 0$, it must be of the form

$$f(x,y,z) = a_1x - a_1y + a_4x^2 + a_5y^2 + a_6z^2 + a_7xy + a_8xz - (2a_6+a_8)yz$$

where (10) holds.

Now consider that f satisfies Property $\Omega 1b_2$, so $\forall \alpha,\beta \in \mathbb{F}$ we have that $f(\alpha,\beta,\alpha)=0$ implies that $\alpha=\beta$. This also cannot be satisfied vacuously by $\Omega 2a_2$. Evaluating yields

$$f(\alpha,\beta,\alpha) = a_1\alpha - a_1\beta + (a_4+a_6+a_8)\alpha^2 + a_5\beta^2 + (a_7-2a_6-a_8)\alpha\beta = 0$$

By (10), $a_4 = a_6 - a_5 - a_7$ which substituting into the last monomial gives that

$$f(\alpha,\beta,\alpha) = a_1(\alpha-\beta) + (2a_6+a_8-a_5-a_7)\alpha^2 + a_5\beta^2 + (a_7-2a_6-a_8)\alpha\beta = 0$$

which can be refactored into the form

$$\begin{aligned} f(\alpha,\beta,\alpha) &= -a_1(\beta-\alpha) + a_5(\beta-\alpha)(\beta+\alpha) + (a_7-2a_6-a_8)\alpha(\beta-\alpha) \\ &= (\beta-\alpha)\underbrace{(-a_1 + a_5[\beta+\alpha] + [a_7-2a_6-a_8]\alpha)}_{\text{trouble factor}} = 0. \end{aligned}$$

Under all values of α and β in $\mathbb{F}$, we need that this equality only holds when $\alpha=\beta$. Thus, we must verify that $\alpha=\beta$ is a valid 0. Substituting $\alpha=\beta$ into the "trouble factor" gives that

$$-a_1 + 2a_5\alpha + (a_7-2a_6-a_8)\alpha = -a_1 + (2a_5+a_7-2a_6-a_8)\alpha = 0$$

for any α. This is only possible if $\mathbf{a_1 = 0}$ and $2a_5+a_7-2a_6-a_8=0$ or $\mathbf{a_8 = 2a_5 + a_7 - 2a_6}$. Plugging this new value in to our original "trouble" expression shows that

$$-a_1 + a_5(\beta+\alpha) + (a_7-2a_6-a_8)\alpha = a_5(\beta+\alpha) - 2a_5\alpha = a_5(\beta-\alpha)$$

which shows that

$$f(\alpha,\beta,\alpha) = a_5(\beta-\alpha)^2$$

This expression is only ever 0 when $\alpha=\beta$ as $a_5 \neq 0$, as desired. So, given $a_5, a_6 \neq 0$, for f to satisfy Properties $\Omega 1a_1, \Omega 1a_2$ and $\Omega 1b_2$, f must be of the form $f(x,y,z) = a_4x^2 + a_5y^2 + a_6z^2 + a_7xy + (2a_5+a_7-2a_6)xz - (2a_5+a_7)yz$ where (10) still holds.
Now consider that f satisfies Property $\Omega 2a_2$, so $\forall \alpha,\beta \in \mathbb{F}\ \exists \gamma \in \mathbb{F}$ such that $f(\alpha,\beta,\gamma)=0$. Substituting gives

$$f(\alpha,\beta,\gamma) = a_4\alpha^2 + a_5\beta^2 + a_6\gamma^2 + a_7\alpha\beta + (2a_5+a_7-2a_6)\alpha\gamma - (2a_5+a_7)\beta\gamma = 0.$$

We can rearrange this expression into the form of a quadratic on γ with coefficients in $\mathbb{F}[\alpha,\beta]$ to get that

$$a_6\gamma^2 + ([2a_5 + a_7 - 2a_6]\alpha - [2a_5 + a_7]\beta)\gamma + a_4\alpha^2 + a_5\beta^2 + a_7\alpha\beta = 0.$$

A γ will exist so long as the discriminant is square, i.e. if for any $\alpha, \beta \in \mathbb{F}$, the expression

$$([2a_5 + a_7 - 2a_6]\alpha - [2a_5 + a_7]\beta)^2 - 4a_6(a_4\alpha^2 + a_5\beta^2 + a_7\alpha\beta)$$

is a square. Refactoring like terms with respect to α and β gives

$$\big([2a_5+a_7-2a_6]^2-4a_4a_6\big)\alpha^2+\big([2a_5+a_7]^2-4a_5a_6\big)\beta^2-2\big([2a_5+a_7-2a_6][2a_5+a_7]+2a_6a_7\big)\alpha\beta$$

Rearranging (10) gives that $a_7 = a_6 - a_4 - a_5$, so substituting that in gives

$$\begin{aligned}\big([a_5 - a_6 - a_4]^2 - 4a_4a_6\big)\alpha^2 + \big([a_5 + a_6 - a_4] - 4a_5a_6\big)\beta^2\\ - 2\big([a_5 - a_6 - a_4][a_5 + a_6 - a_4] + 2a_6[a_6 - a_5 - a_4]\big)\alpha\beta\end{aligned}$$

which simplifies to

$$\big([a_4 - a_5 - a_6]^2 - 4a_5a_6\big)\big(\alpha - \beta\big)^2.$$

Thus, a γ will exist so long as $(a_4 - a_5 - a_6)^2 - 4a_5a_6$ is a square in $\mathbb{F}$. However, factoring $f(\alpha,\beta,\gamma)$ is fundamentally the same as factoring $f(x,y,z)$. Thus, given $f(x,y,z) = 0$, or

$$a_6z^2 + ([2a_5 + a_7 - 2a_7]x - [2a_5 + a_7]y)z + a_4x^2 + a_5y^2 + a_7xy = 0$$

and $(a_4 - a_5 - a_6)^2 - 4a_5a_6 = \omega^2$ for some $\omega \in \mathbb{F}$, we can use the quadratic formula and simplify just as before to get that

$$\begin{aligned} z &= \frac{(2a_5 + a_7)y - (2a_5 + a_7 - 2a_6)x \pm \sqrt{\omega^2(x-y)^2}}{2a_6} \\ &= \frac{(a_5 + a_6 - a_4)y + (a_4 + a_6 - a_5)x \pm \omega(x-y)}{2a_6} \end{aligned}$$

This means that $f(x,y,z)$ is equal to

$$\begin{aligned} f(x,y,z) &= a_6\Big(z - \frac{[a_4 + a_6 - a_5 + \omega]x + [a_5 + a_6 - a_4 - \omega]y}{2a_6}\Big) \\ &\qquad\times\Big(z - \frac{[a_4 + a_6 - a_5 - \omega]x + [a_5 + a_6 - a_4 + \omega]y}{2a_6}\Big) \\ &= \frac{1}{4a_6}\Big([a_4 + a_6 - a_5 + \omega]x + [a_5 + a_6 - a_4 - \omega]y - 2a_6z\Big) \\ &\qquad\times\Big([a_4 + a_6 - a_5 - \omega]x + [a_5 + a_6 - a_4 - \omega]y - 2a_6z\Big) \end{aligned}$$

showing that f factors. Thus, there is no irreducible quadratic f with $a_5, a_6 \neq 0$ such that f is in a strong pair.

In all 4 cases, f factored into a product of linear terms, showing that f cannot be an irreducible quadratic. Thus, there is no irreducible quadratic polynomial over a field of characteristic not 2 in a strong pair. $\square$

Corollary 13 *Let $\mathbb{F}$ be a field with characteristic not 2. Then there is no strong pair of polynomials (f, g) which has either f or g as an irreducible quadratic over $\mathbb{F}$.*

5 Automorphism Classes

In this section, we consider how automorphisms φ of our field of labels $\mathbb{F}$ affects $(g, f)_{\mathbb{F}}$ colorings. Additionally, noting that automorphisms of $\mathbb{F}$ extend to automorphisms of $\mathbb{F}[x, y, z]$ by applying φ to the coefficients, we develop tools for building new pairs of polynomials (g', f') whose colorability is an invariant. This lets us conclude that the family of pairs of polynomials forming a knot invariant features polynomials of unbounded degrees.

First, note that given an automorphism φ of $\mathbb{F}$ and a polynomial $p(x, y, z)$ over $\mathbb{F}$, we get another polynomial $p'(x, y, z)$ over $\mathbb{F}$ obtained by applying φ to every coefficient of p. This then tells us that, treating p and p' as functions from $\mathbb{F}^3$ to $\mathbb{F}$, $\varphi \circ p = p' \circ (\varphi \times \varphi \times \varphi)$. However, the roots of $\varphi \circ p$ are exactly the roots of p, since φ is injective. This then gives us a bijection between the roots of p' and the roots of p, that bijection being exactly $\varphi \times \varphi \times \varphi$. This observation leads us to the following theorem.

Theorem 14 *Let $\mathbb{F}$ be a field and let $f, g \in \mathbb{F}[x, y, z]$ be polynomials such that $(g, f)_{\mathbb{F}}$ colorability is a knot invariant. Then for any field automorphism φ of $\mathbb{F}$, if we define f' and g' to be the polynomials obtained by the process above, then $(g', f')_{\mathbb{F}}$ colorability is a knot invariant.*

Proof. The proof follows in two steps. First, we will show that a knot projection admits a $(g, f)_{\mathbb{F}}$ coloring if and only if it admits a $(g', f')_{\mathbb{F}}$ coloring, using the bijection of roots described above. Then we will use this to demonstrate $(g', f')_{\mathbb{F}}$ colorability is a knot invariant.

Let φ be an automorphism of $\mathbb{F}$, let $f, g \in \mathbb{F}[x, y, z]$ such that $(g, f)_{\mathbb{F}}$ colorability is a knot invariant, and let f' and g' be as defined above. Let ρ be a projection of an oriented knot admitting a valid $(g, f)_{\mathbb{F}}$ coloring, meaning there is an assignment of an element in $\mathbb{F}$ to each strand of ρ such that at least two distinct labels are used and at each left-handed crossing, $g(x, y, z) = 0$, and at each right-handed crossing, $f(x, y, z) = 0$, where x refers to the label on the overstrand, y to the label on the incoming understrand, and z to the label on the outgoing understrand. By the previous observation, $f(x, y, z) = 0$ if and only if $f'(\varphi(x), \varphi(y), \varphi(z)) = 0$ with the analogous condition holding for g and g', so consider the labelling of the strands of ρ obtained by replacing their labels by their images under φ. This labelling is then a $(g', f')_{\mathbb{F}}$ coloring. This coloring is valid as each crossing satisfies the proper equation depending on the handedness, and because φ is an automorphism it is injective, so given distinct labels α and β we see $\varphi(\alpha) \neq \varphi(\beta)$. Hence, at least two distinct labels are used. Thus, if ρ admits a $(g, f)_{\mathbb{F}}$ coloring, then it admits a $(g', f')_{\mathbb{F}}$ coloring. The converse follows immediately as if φ is an automorphism, then φ^{-1} is an automorphism and f is then the result of applying φ^{-1} to the coefficents of f', with the analogous statement for g and g'. So by repeating the argument previously given, we see if ρ admits a valid $(g', f')_{\mathbb{F}}$ coloring then it admits a valid $(g, f)_{\mathbb{F}}$ coloring.

Now, $(g, f)_{\mathbb{F}}$ colorability is a knot invariant if and only if given any oriented knot K and knot projection ρ of K, if ρ' is an oriented knot projection that differs from ρ by the application of a single Reidemeister move, then ρ admits a $(g, f)_{\mathbb{F}}$ coloring if and only if ρ' does as well. But by the previous part, all projections of an oriented knot that admit a $(g, f)_{\mathbb{F}}$ coloring admit a $(g', f')_{\mathbb{F}}$ coloring, and so the previous statement implies ρ is $(g', f')_{\mathbb{F}}$ colorable if and only if ρ' is $(g', f')_{\mathbb{F}}$ colorable. Hence, $(g', f')_{\mathbb{F}}$ colorability is a knot invariant. $\square$

If we restrict our attention to fields $\mathbb{F}$ of characteristic p for some prime number p, then there is a family of nice automorphisms which can be described as mapping polynomials into polynomials. They are all obtained by taking powers of the map $\varphi : \mathbb{F} \to \mathbb{F}$ such that $\varphi(a) = a^p$. This map is called the Fröbenius map, and a proof of its stated properties can be found in Lang [4]. Since composing polynomials together is a polynomial, we get that for any polynomial $p \in \mathbb{F}[x, y, z]$, $p \circ (\varphi \times \varphi \times \varphi)$ is another polynomial whose roots are in bijection with those of p.

Corollary 15 *Let $\mathbb{F}$ be a field of characteristic $p \neq 0$, and let $f, g \in \mathbb{F}[x, y, z]$ be polynomials over $\mathbb{F}$ such that $(g, f)_{\mathbb{F}}$ colorability is a knot invariant. Then for $\varphi(a) := a^p$, the polynomials $f''(x, y, z) = f(\varphi(x), \varphi(y), \varphi(z))$ and $g''(x, y, z) = g(\varphi(x), \varphi(y), \varphi(z))$ have (g'', f'') colorability as a knot invariant.*

Proof. Let $\mathbb{F}$, φ, f, g, f'', and g'' be as defined. Then, there is a bijection between the roots of f and the roots of f'' obtained by mapping $(\alpha, \beta, \gamma) \mapsto (\varphi^{-1}(\alpha), \varphi^{-1}(\beta), \varphi^{-1}(\gamma))$. The same map also defines a bijection between the roots of g and the roots of g''. Hence, any $(g, f)_{\mathbb{F}}$ coloring defines a $(g'', f'')_{\mathbb{F}}$ coloring by taking each label in the $(g, f)_{\mathbb{F}}$ coloring and replacing it by φ^{-1}, and the converse holds by replacing φ^{-1} by φ. Hence by mimicking the proof of Theorem 14, we see that if $(g, f)_{\mathbb{F}}$ colorability is a knot invariant, then so too is $(g'', f'')_{\mathbb{F}}$ colorability. $\square$

Example 16 *Consider a field of characteristic 3, and note by the linear case $f(x, y, z) = g(x, y, z) = 2x - y - z$ has $(g, f)_{\mathbb{F}}$ colorability as a knot invariant. The Fröbenius map is then $\sigma(a) = a^3$, and so composing on the right gives $f'(x, y, z) = g'(x, y, z) = 2x^3 - y^3 - z^3$. By the corollary we see that $(g', f')_{\mathbb{F}}$ colorability is also a knot invariant. We can then apply the corollary again to get the polynomial $f''(x, y, z) = g''(x, y, z) = 2x^9 - y^9 - z^9$ has $(g'', f'')_{\mathbb{F}}$ colorability as a knot invariant.*

We see from this example that we can recursively apply this corollary to get polynomials of unbounded degree. We conclude this in the following corollary.

Corollary 17 *Let $\mathbb{F}$ be a field of nonzero characteristic. Then there exists an infinite family of pairs of polynomials over $\mathbb{F}$ whose colorability forms a knot invariant. Additionally, this family has no upper bound on the degrees of its polynomials.*

Proof. Let $\mathbb{F}$ have characteristic p, where p is some prime number. Then consider polynomials of the form $f_n(x, y, z) = 2x^{p^n} - y^{p^n} - z^{p^n}$. We claim that for any positive integer n, $(f_n, f_n)_{\mathbb{F}}$ colorability is a knot invariant. However, this follows immediately by induction as when $n = 1$, $f_1(x, y, z) = 2x - y - z$ is the Fox-p colorability polynomial, and thus $(f_1, f_1)_{\mathbb{F}}$ colorability is a knot invariant. This also follows from the established linear case, so we have established the base case for induction. However, if $(f_n, f_n)_{\mathbb{F}}$ colorability is a knot invariant, then note that

$$f_{n+1}(x, y, z) = 2x^{p^{n+1}} - y^{p^{n+1}} - z^{p^{n+1}} = 2(x^{p^n})^p - (y^{p^n})^p - (z^{p^n})^p$$

shows that f_{n+1} is obtained by applying the Fröbenius map to each coordinate of f_n. Hence by the corollary $(f_{n+1}, f_{n+1})_{\mathbb{F}}$ colorability is also a knot invariant. Each f_n is distinct, and hence there is an infinite number of polynomials over $\mathbb{F}$ whose colorability forms a knot invariant over $\mathbb{F}$. Additionally, each f_n has degree p^n, and so the polynomials forming a knot invariant have no upper bound on their degrees. □

6 Conclusion

Recall, $(g, f)_{\mathbb{F}}$ colorings arose from generalizing Fox n-colorings in an attempt to produce new knot invariants. One goal for further work could be to search for more strong pairs of polynomials. Additionally, it would be desirable to find polynomials that provide invariants which distinguish more knots than the Fox n-coloring does. In particular, the Fox n-coloring does not distinguish torus knots from the unknot, so one might seek a pair of strong polynomials that can color a torus knot.

The properties used to define strong pairs of polynomials come from a naive approach to translate the Reidemeister moves into algebraic conditions that the polynomials must satisfy. It is possible that these properties are too restrictive, and that there are pairs of non-strong polynomials with colorability that is a knot invariant. One approach to studying this larger class of polynomials could be to use algebraic geometry. With an algebraic geometry approach, it may be possible to find a deeper geometric explanation for the restrictiveness of our properties, or more ambitiously find less restrictive algebraic conditions which maintain colorability as a knot invariant.

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