

# Hurwitz Orbits of Equal Size

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**Abstract** - We provide a variety of cases in which two factorizations have Hurwitz orbits of the same size. We begin with prototypical results about factorizations of length two, and show that cycling elements or flipping and inverting elements in any factorization preserves Hurwitz orbit size. We prove that “double reverse” factorizations in groups with special presentations have Hurwitz orbits of equal size, and end with applications to complex reflection groups.

**Keywords** : Hurwitz action; Hurwitz orbit; factorizations of elements in groups; complex reflection groups; Coxeter groups; Shephard groups

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## 1 Introduction

The Hurwitz action is a combinatorial action on tuples of group elements, based on basic moves that swap two elements, modifying one, while preserving the product. Its study dates to work of Hurwitz in the late 19th century. This paper is motivated by an observation about the Hurwitz action and the complex reflection group  $G_6 = \langle a, b \mid a^3 = b^2 = 1, ababab = bababa \rangle$ . We took factorizations containing only  $a$ ,  $b$ , and  $a^{-1}$ , and randomly permuted the factors. Permuting elements in these factorizations preserved the size of their Hurwitz orbits (see Conjecture 7.1). This observation raised the question “In what ways can we reorder or modify the factors in a factorization while preserving the size of its Hurwitz orbit?”

In Section 3, we prove that for elements  $x$  and  $y$  in a group  $G$ , the Hurwitz orbits of  $(x, y)$ ,  $(x^{-1}, y^{-1})$ , and  $(y, x)$  have equal size. We then generalize these results to longer factorizations in several ways. We prove in Section 4 that for any  $x_1, x_2, \dots, x_\ell$  in a group  $G$ ,  $(x_1, x_2, \dots, x_\ell)$  and  $(x_2, \dots, x_\ell, x_1)$  have Hurwitz orbits of equal size. In Section 5 we show that  $(x_\ell^{-1}, \dots, x_2^{-1}, x_1^{-1})$  belongs to an orbit of the same size as well. In the special case where all  $x_1, x_2, \dots, x_\ell$  have order 1 or 2, we see that the Hurwitz orbit of  $(x_1, x_2, \dots, x_\ell)$  has the same size as the orbit of  $(x_\ell, x_{\ell-1}, \dots, x_1)$ .

In Section 6, we prove that  $(x_1, x_2, \dots, x_\ell)$  and  $(x_\ell, x_{\ell-1}, \dots, x_1)$  have Hurwitz orbits of equal size when the  $x_i$  belong to a generating set in a group presentation with “reversible relations”. We end in Section 7 with applications to Coxeter groups and Shephard groups, as well as a conjecture about  $G_6$ .



## 2 Background

In a group  $G$ , a *factorization* of an element  $g \in G$  is a tuple of elements in  $G$  that multiply to  $g$ . Given a factorization  $(x_1, x_2, \dots, x_\ell)$  of an element in  $G$ , a *Hurwitz move* at position  $i$ , with  $1 \leq i \leq \ell - 1$ , is defined to be the following operation:

$$\begin{aligned} & (x_1, \dots, x_{i-1}, \quad x_i, \quad x_{i+1}, \quad x_{i+2}, \dots, x_\ell) \xrightarrow{\sigma_i} \\ & (x_1, \dots, x_{i-1}, \quad x_{i+1}, \quad x_{i+1}^{-1}x_i x_{i+1}, \quad x_{i+2}, \dots, x_\ell). \end{aligned}$$

The inverse  $\sigma_i^{-1}$  of the Hurwitz move  $\sigma_i$  is given by

$$\begin{aligned} & (x_1, \dots, x_{i-1}, \quad x_i, \quad x_{i+1}, \quad x_{i+2}, \dots, x_\ell) \xrightarrow{\sigma_i^{-1}} \\ & (x_1, \dots, x_{i-1}, \quad x_i x_{i+1} x_i^{-1}, \quad x_i, \quad x_{i+2}, \dots, x_\ell). \end{aligned}$$

The *Hurwitz orbit* of a factorization  $T$  is the set of factorizations which can be produced by applying Hurwitz moves to  $T$ . We use the term “orbit” because Hurwitz moves give rise to an action of the braid group. The term *size* refers to the cardinality of the Hurwitz orbit.

The first paper to study the Hurwitz action was written by A. Hurwitz in 1891 [7]. Historically, researchers who have studied the Hurwitz action have been interested in when two factorizations belong to the same Hurwitz orbit. For example, [9] and [2] answered the question of when two transposition factorizations in the symmetric group belong to the same Hurwitz orbit. [6], [15], and [3] answered this same question, instead for factorizations in the dihedral group. Many authors have also considered the case of factorizations of special elements in complex reflection groups or Coxeter groups [4, 10, 11, 1, 8, 14, 5]. Researchers have explored the Hurwitz action from different perspectives as well. In 2019, Mühle and Ripoll studied the Hurwitz action as a connectivity property on posets [12]. Unlike previous papers, our paper is concerned not with the question of when two factorizations belong to the same Hurwitz orbit, but instead when they have Hurwitz orbits of the same size.

## 3 Factorizations of length two

In this section we prove that  $(x, y)$ ,  $(y, x)$ , and  $(x^{-1}, y^{-1})$  have Hurwitz orbits of the same size. This observation on the orbit sizes of length-2 factorizations serves as a prototype and motivating example for the more general results in later sections. We begin with a lemma on the general form of  $(x, y)$  after any number of Hurwitz moves.

**Lemma 3.1** *Let  $G$  be a group,  $n$  an integer, and  $x, y \in G$ . Then*

1.  $\sigma^{2n}(x, y) = (y^{-1}(x^{-1}y^{-1})^{n-1}x(yx)^{n-1}y, (y^{-1}x^{-1})^n y(xy)^n)$  and
2.  $\sigma^{2n+1}(x, y) = ((y^{-1}x^{-1})^n y(xy)^n, y^{-1}(x^{-1}y^{-1})^n x(yx)^n y)$ .



**Proof.**

The proof of (a) is by induction on  $n$ . When  $n = 0$ ,

$$(y^{-1}(x^{-1}y^{-1})^{-1}x(yx)^{-1}y, (y^{-1}x^{-1})^0y(xy)^0) = (y^{-1}(yx)x(x^{-1}y^{-1})y, y) = (x, y)$$

as required. Now assume the lemma holds when  $n = k$ , and let  $(a, b) = \sigma^{2k}(x, y)$ . By assumption,  $b = (y^{-1}x^{-1})^k y(xy)^k$ . Perform two Hurwitz moves on  $(a, b)$  to obtain

$$\sigma^{2(k+1)}(x, y) = (b^{-1}ab, b^{-1}a^{-1}bab).$$

Since Hurwitz moves on a factorization preserve its product,  $ab = xy$  and  $b^{-1}a^{-1} = y^{-1}x^{-1}$ . By assumption  $b = (y^{-1}x^{-1})^k y(xy)^k$ . Thus

$$\begin{aligned} \sigma^{2(k+1)}(x, y) &= (b^{-1}xy, y^{-1}x^{-1}bxy) \\ &= ((y^{-1}x^{-1})^k y^{-1}(xy)^k xy, y^{-1}x^{-1}(y^{-1}x^{-1})^k y(xy)^k xy) \\ &= (y^{-1}(x^{-1}y^{-1})^{(k+1)-1}x(yx)^{(k+1)-1}y, (y^{-1}x^{-1})^{k+1}y(xy)^{k+1}), \end{aligned}$$

and the induction is complete.

Now let  $(a, b) = \sigma^{2n+1}(x, y)$ . By part (a),  $a = (y^{-1}x^{-1})^n y(xy)^n$ . Since  $ab = xy$ ,  $b = (y^{-1}x^{-1})^n y^{-1}(xy)^n xy$ . Thus

$$(a, b) = ((y^{-1}x^{-1})^n y(xy)^n, y^{-1}(x^{-1}y^{-1})^n x(yx)^n y).$$

The proof for negative  $n$  is similar. □

**Theorem 3.2** *Let  $G$  be any group, and  $x, y$  any elements of  $G$ . The Hurwitz orbit of the factorization  $(x, y)$  has the same size as the Hurwitz orbit of  $(y, x)$ .*

A version of this result appears as Lemma 2.9 in [12].

**Proof.** Let  $n$  be a nonnegative integer, and suppose  $\sigma^{2n}(x, y) = (x, y)$ . By Lemma 3.1,

$$(x, y) = (y^{-1}(x^{-1}y^{-1})^{n-1}x(yx)^{n-1}y, (y^{-1}x^{-1})^n y(xy)^n).$$

Observe that the second factor of this factorization may be written as

$$y^{-1}(x^{-1}y^{-1})^{n-1}x^{-1}(yx)^n y,$$

so that

$$y = (x^{-1}y^{-1})^{n-1}x^{-1}(yx)^n = x^{-1}(y^{-1}x^{-1})^{n-1}y(xy)^{n-1}x.$$

By Lemma 3.1,

$$\begin{aligned} \sigma^{2n}(y, x) &= (x^{-1}(y^{-1}x^{-1})^{n-1}y(xy)^{n-1}x, (x^{-1}y^{-1})^n x(yx)^n) \\ &= (y, (x^{-1}y^{-1})^n x(yx)^n). \end{aligned}$$

The product of this factorization is  $yx$ , so the right-hand value of this factorization is  $x$ , and  $\sigma^{2n}(y, x) = (y, x)$ .



If  $\sigma^{2n+1}(x, y) = (x, y)$ , then by Lemma 3.1

$$y = y^{-1}(x^{-1}y^{-1})^n x(yx)^n y = (x^{-1}y^{-1})^n x(yx)^n. \quad (1)$$

By Lemma 3.1 and (1)

$$\begin{aligned} \sigma^{2n+1}(y, x) &= ((x^{-1}y^{-1})^n x(yx)^n, x^{-1}(y^{-1}x^{-1})^n y(xy)^n x) \\ &= (y, x^{-1}(y^{-1}x^{-1})^n y(xy)^n x) \end{aligned}$$

So  $\sigma^{2n+1}(y, x) = (y, x)$

We have shown that for any nonnegative integer  $n$ ,  $\sigma^n(x, y) = (x, y)$  implies  $\sigma^n(y, x) = (y, x)$ . Thus the size of the Hurwitz orbit of  $(y, x)$  is less than or equal to the size of the Hurwitz orbit of  $(x, y)$ . By interchanging  $x$  and  $y$  in the argument, it can be seen that  $\sigma^n(y, x) = (y, x)$  implies  $\sigma^n(x, y) = (x, y)$ . So  $(x, y)$  and  $(y, x)$  have Hurwitz orbits of the same size.  $\square$

We expand on Lemma 3.1 to prove that  $(x, y)$  has the same sized Hurwitz orbit as  $(x^{-1}, y^{-1})$ .

**Theorem 3.3** *Let  $G$  be any group, and  $x, y$  any elements of  $G$ . The Hurwitz orbit of the factorization  $(x, y)$  has the same size as the Hurwitz orbit of  $(x^{-1}, y^{-1})$ .*

**Proof.**

Let  $n$  be a nonnegative integer, and assume  $\sigma^{2n}(x, y) = (x, y)$ . By Lemma 3.1,  $y = (y^{-1}x^{-1})^n y(xy)^n$ , so  $(xy)^n y = y(xy)^n$ . Conjugate on both sides by  $y^{-1}$  to obtain

$$\begin{aligned} y(xy)^n y y^{-1} &= y y(xy)^n y^{-1} \\ (yx)^n y &= y(yx)^n. \end{aligned}$$

Multiplying this last equality on both sides by  $y^{-1}$ , we have that  $y^{-1}$  and  $(yx)^n$  commute. So  $y^{-1} = (yx)^n y^{-1} (x^{-1}y^{-1})^n$ , which is precisely the right-hand value of  $\sigma^{2n}(x^{-1}, y^{-1})$  by Lemma 3.1. Thus  $\sigma^{2n}(x^{-1}, y^{-1}) = (x^{-1}, y^{-1})$ .

If  $\sigma^{2n+1}(x, y) = (x, y)$ , then Lemma 3.1 gives

$$y = y^{-1}(x^{-1}y^{-1})^n x(yx)^n y,$$

so that  $y^{-1} = y^{-1}(x^{-1}y^{-1})^n x^{-1}(yx)^n y = (x^{-1}y^{-1})^n x^{-1}(yx)^n$ , or equivalently  $x^{-1} = (yx)^n y^{-1}(x^{-1}y^{-1})^n$ . But, by Lemma 3.1, this implies the left-hand value of  $\sigma^{2n+1}(x^{-1}, y^{-1})$  is  $x^{-1}$ , and so necessarily the right-hand value must be  $y^{-1}$ .

Altogether we have shown that  $\sigma^n(x, y) = (x, y)$  implies  $\sigma^n(x^{-1}, y^{-1}) = (x^{-1}, y^{-1})$ . With little modification the same argument can be used to show that the converse is true as well. Thus  $(x, y)$  and  $(x^{-1}, y^{-1})$  have Hurwitz orbits of the same size.  $\square$

## 4 Cycling

In this section, we show that cycling elements in a factorization leaves Hurwitz orbit size unchanged.



**Lemma 4.1** *Let  $G$  be any group containing elements  $x_1, x_2, \dots, x_\ell$  and  $y$ . Then the factorizations  $(x_1, x_2, \dots, x_\ell)$  and  $(y^{-1}x_1y, y^{-1}x_2y, \dots, y^{-1}x_\ell y)$  have Hurwitz orbits of the same size.*

**Proof.**

To prove that the orbits are of equal size, we will find a bijection between them. First we show that if

$$\sigma_{p_1} \dots \sigma_{p_n}(x_1, x_2, \dots, x_\ell) = (z_1, z_2, \dots, z_\ell),$$

then

$$\sigma_{p_1} \dots \sigma_{p_n}(y^{-1}x_1y, y^{-1}x_2y, \dots, y^{-1}x_\ell y) = (y^{-1}z_1y, y^{-1}z_2y, \dots, y^{-1}z_\ell y),$$

where each  $p_i \in \{1, \dots, \ell - 1\}$ . We verify that this is true when one Hurwitz move is applied: if  $1 \leq i \leq \ell - 1$ , then

$$\begin{aligned} (\dots, x_i, x_{i+1}, \dots) &\xrightarrow{\sigma_i} (\dots, x_{i+1}, x_{i+1}^{-1}x_i x_{i+1}, \dots), \\ (\dots, y^{-1}x_i y, y^{-1}x_{i+1} y, \dots) &\xrightarrow{\sigma_i} (\dots, y^{-1}x_{i+1} y, y^{-1}x_{i+1}^{-1}x_i x_{i+1} y, \dots). \end{aligned}$$

The result follows by induction.

Observe that this if/then relationship goes both ways:

$$Y = (y^{-1}x_1y, y^{-1}x_2y, \dots, y^{-1}x_\ell y)$$

is the factorization obtained by conjugating every element in

$$X = (x_1, x_2, \dots, x_\ell)$$

by  $y$ , and  $X$  is the factorization obtained by conjugating every element in  $Y$  by  $y^{-1}$ .

We may define a map

$$(z_1, z_2, \dots, z_\ell) \mapsto (y^{-1}z_1y, y^{-1}z_2y, \dots, y^{-1}z_\ell y),$$

from the Hurwitz orbit of  $(x_1, x_2, \dots, x_\ell)$  to that of  $(y^{-1}x_1y, y^{-1}x_2y, \dots, y^{-1}x_\ell y)$ . This function is the desired bijection.  $\square$

**Theorem 4.2** *Let  $G$  be any group containing elements  $x_1, x_2, \dots, x_\ell$ . Then the factorizations  $(x_1, x_2, \dots, x_\ell)$  and  $(x_2, \dots, x_\ell, x_1)$  have Hurwitz orbits of the same size.*

**Proof.** Suppose we apply the Hurwitz moves  $\sigma_{\ell-1}, \sigma_{\ell-2}, \dots, \sigma_1$  in that order to the factorization  $(x_2, \dots, x_\ell, x_1)$ . The resulting factorization is  $(x_1, x_1^{-1}x_2x_1, \dots, x_1^{-1}x_\ell x_1)$ . This resulting factorization is identical to  $(x_1, x_2, \dots, x_\ell)$ , except every element has been conjugated by  $x_1$ ; in particular  $x_1 = x_1^{-1}x_1x_1$ . By Lemma 4.1, our theorem is true.  $\square$



## 5 Reverses

In this section we show that  $(x_1, x_2, \dots, x_\ell)$  will always have the same sized Hurwitz orbit as  $(x_\ell^{-1}, \dots, x_2^{-1}, x_1^{-1})$ . We also expand on some ideas presented in Section 4.

**Theorem 5.1** *Let  $G$  be any group containing elements  $x_1, x_2, \dots, x_\ell$ . Then the factorizations  $(x_1, x_2, \dots, x_\ell)$  and  $(x_\ell^{-1}, \dots, x_2^{-1}, x_1^{-1})$  have Hurwitz orbits of the same size.*

**Proof.** To prove that the orbits are of equal size, we will find a bijection between them. First we show that if

$$\sigma_{p_1} \dots \sigma_{p_n}(x_1, x_2, \dots, x_\ell) = (z_1, z_2, \dots, z_\ell),$$

then

$$\sigma_{\ell-p_1}^{-1} \dots \sigma_{\ell-p_n}^{-1}(x_\ell^{-1}, \dots, x_2^{-1}, x_1^{-1}) = (z_\ell^{-1}, \dots, z_2^{-1}, z_1^{-1})$$

where each  $p_i \in \{1, \dots, \ell - 1\}$ . We verify that this is true when one Hurwitz move is applied: if  $1 \leq i \leq \ell - 1$ , then

$$\begin{aligned} (\dots, x_i, x_{i+1}, \dots) &\xrightarrow{\sigma_i} (\dots, x_{i+1}, x_{i+1}^{-1}x_i x_{i+1}, \dots), \\ (\dots, x_{i+1}^{-1}, x_i^{-1}, \dots) &\xrightarrow{\sigma_{\ell-i}^{-1}} (\dots, x_{i+1}^{-1}x_i^{-1}x_{i+1}, x_{i+1}^{-1}, \dots). \end{aligned}$$

The result follows by induction. In fact this if/then relationship goes both ways because  $(x_k^{-1})^{-1} = x_k$ .

We may define the following function: some factorization  $(z_1, z_2, \dots, z_\ell)$  in the Hurwitz orbit of  $(x_1, x_2, \dots, x_\ell)$  is mapped to the factorization  $(z_\ell^{-1}, \dots, z_2^{-1}, z_1^{-1})$ , which is in the Hurwitz orbit of  $(x_\ell^{-1}, \dots, x_2^{-1}, x_1^{-1})$ . This function is the desired bijection.  $\square$

One interesting case of Theorem 5.1 is when every element in a factorization has order 1 or 2.

**Corollary 5.2** *Let  $G$  be any group containing  $x_1, x_2, \dots, x_\ell$  such that  $x_k$  has order 1 or 2 for  $k = 1, \dots, \ell$ . Then  $(x_1, x_2, \dots, x_\ell)$  and  $(x_\ell, \dots, x_2, x_1)$  have Hurwitz orbits of the same size.*

**Proof.** If  $x_k$  has order 1 or 2, then  $x_k = x_k^{-1}$ . So the factorization  $(x_\ell^{-1}, \dots, x_2^{-1}, x_1^{-1})$  is equal to  $(x_\ell, \dots, x_2, x_1)$ . Therefore, by Theorem 5.1 this corollary is true.  $\square$

**Remark 5.3** In Theorem 4.2 we prove that the factorizations

$$(x_1, x_2, \dots, x_\ell), (x_2, \dots, x_\ell, x_1)$$

will always have the same sized Hurwitz orbits. This theorem produces particularly interesting results when observing factorizations containing only two distinct elements.

Suppose we have a factorization  $(x_1, x_2, \dots, x_\ell)$  such that each  $x_i$ ,  $1 \leq i \leq \ell$ , is equal to  $y$  or  $z$ . One can show that if the factorization has length  $\ell$ ,  $\ell \leq 5$ , then  $(x_1, x_2, \dots, x_\ell)$  has



the same sized Hurwitz orbit as its reverse,  $(x_\ell, \dots, x_2, x_1)$ . This is because  $(x_\ell, \dots, x_2, x_1)$  can be obtained by cycling  $(x_1, x_2, \dots, x_\ell)$ .

The following is an example. Suppose we want to show that  $(z, z, y, z, y)$  has the same sized Hurwitz orbit as  $(y, z, y, z, z)$ . Using Theorem 4.2, we can show that  $(z, z, y, z, y)$  has the same sized Hurwitz orbit as  $(z, y, z, y, z)$  which has the same sized Hurwitz orbit as  $(y, z, y, z, z)$ .

However, this property does not necessarily hold when there are more than two distinct elements in the factorization, or when the length of the factorization is greater than 5. For example, suppose you take the factorization  $(x, y, z)$  and you want to prove that its Hurwitz orbit has the same size as its reverse,  $(z, y, x)$ . Cycling, we get  $(x, y, z) \approx (y, z, x) \approx (z, x, y) \approx (x, y, z)$ . We have cycled completely through  $(x, y, z)$  without producing  $(z, y, x)$ . Now let us take a look at a case where the factorization has length greater than 5. Let us take the factorization  $(x, x, y, x, y, y)$  and show that we cannot produce its reverse  $(y, y, x, y, x, x)$  through cycling:

$$\begin{aligned} (x, x, y, x, y, y) &\approx (x, y, x, y, y, x) \approx (y, x, y, y, x, x) \approx \\ &(x, y, y, x, x, y) \approx (y, y, x, x, y, x) \approx (y, x, x, y, x, y) \approx (x, x, y, x, y, y). \end{aligned}$$

We have shown a case of a length 6 factorization where cycling does not produce its reverse. Our remark proves that any factorization of length  $\ell \leq 5$ , with 1 or 2 distinct elements has the same sized Hurwitz orbit as its reverse.

## 6 Double reverses and reverse relations

In this section, we investigate groups with presentations in which relations between the generators are reversible. We prove that in such groups, Hurwitz orbit size is preserved when reversing factorizations of elements from the generating set. This result follows as a corollary of the main theorem in this section, Theorem 6.6, which proves that a modified reversal operation on factorizations consisting of arbitrary words preserves Hurwitz orbit size.

We begin by recalling a number of standard definitions from the literature – see, for example, [13]. Let  $X = \{x_1, x_2, \dots, x_n\}$  be a set of distinct elements, and  $X^{-1} = \{x_1^{-1}, x_2^{-1}, \dots, x_n^{-1}\}$  a set of elements distinct from each other and from the elements of  $X$ . A *word* on  $X \cup X^{-1}$  is a string of finitely many elements, or *letters*, from  $X \cup X^{-1}$ . An *inverse pair* is a word of the form  $x_i x_i^{-1}$  or  $x_i^{-1} x_i$ , and a word is said to be *reduced* if it contains no inverse pairs. The set  $F(X)$  of all reduced words on  $X \cup X^{-1}$  is a group under the operation of concatenation followed by deletion of inverse pairs; it is called the *free group on  $n$  generators*. The inverse of an element  $x_{i_1} x_{i_2} \dots x_{i_p} \in F(X)$  is  $x_{i_p}^{-1} x_{i_{p-1}}^{-1} \dots x_{i_1}^{-1}$ .

Now let  $G$  be a group and  $S$  a subset of  $G$ . The *normal closure* of  $S$  in  $G$  is defined as the intersection of all normal subgroups of  $G$  containing  $S$ . Every element of the normal closure of  $S$  in  $G$  is the product of conjugates of elements of  $S$ .

A *presentation*  $\langle X \mid R \rangle$  of a finitely generated group  $G$  consists of a generating set  $X = \{x_1, x_2, \dots, x_n\}$  and a set of *relations*  $R \subset F(X)$ , such that the quotient of  $F(X)$



by the normal closure of  $R$  in  $F(X)$  is isomorphic to  $G$ . We denote the normal closure of  $R$  in  $F(X)$  by  $N$ .

The set  $X$  is identified with a generating set for  $G$ , and two elements  $x_{i_1}x_{i_2}\dots x_{i_p}$  and  $x_{j_1}x_{j_2}\dots x_{j_q}$  are equal in  $G$  if and only if they belong to the same coset of  $N$  in  $F(X)$ . To distinguish between the two groups, it will be useful to write  $a = b$  for equality in  $F(X)$  and  $a \equiv b$  for equality in  $G$ .

If  $a \in F(X)$ , the *reverse*  $a^*$  of  $a$  is obtained by reversing the order of the letters in  $a$ . If  $a = x_{i_1}x_{i_2}\dots x_{i_p}$ , then  $a^* = x_{i_p}x_{i_{p-1}}\dots x_{i_1}$ . If some inverse pair  $x_i x_i^{-1}$  appears in  $a^*$ , then  $x_i^{-1} x_i$  appears in  $a$ , a contradiction because  $a$  is reduced. So the reverse of a reduced word is itself a reduced word. With this notation we have

$$(a^{-1})^* = x_{i_1}^{-1} x_{i_2}^{-1} \dots x_{i_p}^{-1} = (a^*)^{-1}. \quad (2)$$

We begin with a proposition on how to calculate reverses of products in  $F(X)$ .

**Proposition 6.1** *Let  $a, a_1, a_2, \dots, a_m$  be elements of  $F(X)$  such that  $a = a_1 a_2 \dots a_m$ . Then  $a^* = a_m^* a_{m-1}^* \dots a_1^*$ . That is, multiplying words in  $F(X)$  and reversing their product is equivalent to reversing each word, reversing the order of multiplication, and taking the product.*

**Proof.** It suffices to verify the statement when  $a$  is a product of two elements  $a_1$  and  $a_2$  of  $F(X)$ . The proposition then follows by induction.

Suppose  $a = x_{i_1}x_{i_2}\dots x_{i_p}$ ,  $a_1 = x_{j_1}x_{j_2}\dots x_{j_q}$ , and  $a_2 = x_{k_1}x_{k_2}\dots x_{k_r}$ . We want to show  $x_{i_p}x_{i_{p-1}}\dots x_{i_1} = (x_{k_r}x_{k_{r-1}}\dots x_{k_1})(x_{j_q}x_{j_{q-1}}\dots x_{j_1})$ . Since  $a_1$  and  $a_2$  are reduced, either  $x_{k_r}x_{k_{r-1}}\dots x_{k_1}x_{j_q}x_{j_{q-1}}\dots x_{j_1}$  is reduced or  $x_{k_1}x_{j_q}$  is an inverse pair. In the first case it follows that  $x_{j_1}x_{j_2}\dots x_{j_q}x_{k_1}x_{k_2}\dots x_{k_r}$  is reduced and is thus equal to  $a$ . So  $a^* = x_{k_r}x_{k_{r-1}}\dots x_{k_1}x_{j_q}x_{j_{q-1}}x_{j_1} = a_2^* a_1^*$ , as required.

If  $x_{k_1}x_{j_q}$  is an inverse pair, then so is its reverse. Deleting these inverse pairs gives

$$\begin{aligned} a &= (x_{j_1}x_{j_2}\dots x_{j_{q-1}})(x_{k_2}x_{k_3}\dots x_{k_r}), \\ a_2^* a_1^* &= (x_{k_r}x_{k_{r-1}}\dots x_{k_2})(x_{j_{q-1}}x_{j_{q-2}}\dots x_{j_1}). \end{aligned}$$

This process continues until there are no more inverse pairs to be deleted. □

The groups of interest in this section can now be defined. A presentation  $\langle X \mid R \rangle$  for a group  $G$  is said to have *reversible relations* if for every  $x_{i_1}x_{i_2}\dots x_{i_p} \in R$  the reverse word  $x_{i_p}x_{i_{p-1}}\dots x_{i_1}$  is contained in  $N$ .

For example, two commonly used presentations of the dihedral group of order  $2n$  have reversible relations. Let  $D_{2n} = \langle r, s \mid r^n = s^2 = 1, rs = sr^{-1} \rangle$ . Formally we can say  $D_{2n} = \langle r, s \mid R \rangle$  where  $R = \{r^n, s^2, rsrs^{-1}\}$ . Then  $(r^n)^* = r^n \in R$ ,  $(s^2)^* = s^2 \in R$ , and  $(rsrs^{-1})^* = s^{-1}rsr = s^{-1}(rsrs^{-1})s \in N$ , so this presentation has reversible relations.

Now consider the presentation  $\langle a, b \mid a^2 = b^2 = (ab)^n = 1 \rangle$  for  $D_{2n}$ . Since the reverse of  $(ab)^n$  is  $(ba)^n = b(ab)^n b^{-1} \in N$ , this presentation again has reversible relations. An important part of our definition of reversible relations is that reverse words are not required to be in the set of relations  $R$  itself, but rather in the normal closure  $N$  of  $R$ .





The presentation  $\langle a, b \mid a^4 = 1, a^2 = b^2, ba = a^{-1}b \rangle$  for the quaternion group  $Q_8$  is another example of a presentation with reversible relations. However,  $Q_8$  can also be given by  $\langle -1, i, j, k \mid (-1)^2 = 1, i^2 = j^2 = k^2 = ijk = -1 \rangle$ , which does not have the same property: while  $ijk(-1) = 1$ , the reverse word  $(-1)kji$  is equal to  $(-1)k(-1)k = k^2 = -1 \neq 1$ .

**Remark 6.2** Suppose  $\langle X \mid R \rangle$  is a group presentation, and we want to show that it has reversible relations. Our definition would have us reverse each word in  $R$  and check that these reverses belong to  $N$ . This process can be made easier by the following observations. Relations giving the order of a generator, such as  $r^n = 1$ , are always reversible. Equations of the form  $a = a^*$ , such as  $xyxy = yxyx$ , are reversible. Equations  $a = b$  where  $a$  and  $b$  are *palindromes*, which is to say  $a = a^*$  and  $b = b^*$  in  $F(X)$ , are reversible.

With an arbitrary presentation, the reverses of two equal words need not be equal. For example, in  $Q_8$  we have  $ij = k$  but  $ji = -k \neq k$ . The next proposition shows that this never happens when reversing words on a generating set with reversible relations.

**Proposition 6.3** *Suppose that the group  $G$  is given by a presentation  $\langle X \mid R \rangle$  with reversible relations. If  $a, b \in F(X)$  and  $a \equiv b$  in  $G$ , then  $a^* \equiv b^*$ .*

**Proof.** The elements  $a$  and  $b$  belong to the same coset of  $N$  in  $F(X)$ , so  $a = bn$  for some  $n \in N$ . Proposition 6.1 gives  $a^* = n^*b^*$ , so  $a^*$  and  $b^*$  belong to the same coset of  $N$  if  $n^* \in N$ . That is,  $a^* \equiv b^*$  if  $N$  is closed under the operation of reversing words.

Every element  $n \in N$  is the product of conjugates of elements of  $R$ . If

$$n = b_1 a_1 b_1^{-1} b_2 a_2 b_2^{-1} \cdots b_m a_m b_m^{-1}$$

with  $a_i \in R, b_i \in F(X), i = 1, 2, \dots, m$ , then by Proposition 6.1

$$n^* = (b_m^{-1})^* a_m^* b_m^* (b_{m-1}^{-1})^* a_{m-1}^* b_{m-1}^* \cdots (b_1^{-1})^* a_1^* b_1^*.$$

The presentation  $\langle X \mid R \rangle$  has reversible relations, so  $a_i^* \in N$  for all  $i$ . Since  $(b_i^{-1})^* = (b_i^*)^{-1}$ ,  $n^*$  is the product of conjugates of elements of  $N$ , and thus  $n^* \in N$ .  $\square$

The main theorem in this section concerns an operation on factorizations which does more than just reverse the order of the factors: if  $\langle X \mid R \rangle$  is a presentation for the group  $G$ , and if  $U$  and  $V$  are length- $\ell$  factorizations of elements in  $G$ , we say  $U$  is a *double reverse* of  $V$  if there exist  $a_1, a_2, \dots, a_\ell \in F(X)$  such that

$$\begin{aligned} U &= (a_1, a_2, \dots, a_\ell), \\ V &= (a_\ell^*, a_{\ell-1}^*, \dots, a_1^*). \end{aligned}$$

That is,  $U$  is a double reverse of  $V$  if  $V$  can be obtained by reversing the order in which words appear in  $U$  and reversing each word. This relation is symmetric, so we say  $U$  and  $V$  are double reverses.



**Remark 6.4** Double reverses are not unique. For example, with the presentation for  $Q_8$  given above we have that  $T = (i, j, ij)$  and  $U = (ji, j, i)$  are double reverses. Also  $V = (k, j, i)$  is a double reverse of  $T$  because  $ij = k$ . But  $ji = -k \neq k$ , so  $U \neq V$ .

The next lemma lets us preserve the double reverse relation after performing a series of Hurwitz moves on one factorization. We will use it to build a map between the Hurwitz orbits of double reverse factorizations.

**Lemma 6.5** *Let  $G = \langle X \mid R \rangle$ , let  $U$  and  $V$  be length- $\ell$  factorizations of elements in  $G$ , and assume  $U$  and  $V$  are double reverses. Perform  $n$  Hurwitz moves  $\sigma_{p_1}, \sigma_{p_2}, \dots, \sigma_{p_n}$  on  $U$  in order as listed, with  $p_i \in \{1, \dots, \ell - 1\}$  for  $i = 1, \dots, n$ . Call the factorization produced by these moves  $U_n$ . Let  $V_n$  be the factorization obtained by performing inverse Hurwitz moves  $\sigma_{\ell-p_1}^{-1}, \sigma_{\ell-p_2}^{-1}, \dots, \sigma_{\ell-p_n}^{-1}$  on  $V$ . Then  $U_n$  and  $V_n$  are double reverses.*

**Proof.** Induct on  $n$ ; the base case  $n = 0$  follows from having defined  $U$  and  $V$  as double reverses. Let  $k$  be a nonnegative integer such that the lemma holds when  $n = k$ , and choose integers  $p_1, \dots, p_{k+1}$  between 1 and  $\ell - 1$ . Let

$$\begin{aligned} U_k &= \sigma_{p_k} \sigma_{p_{k-1}} \cdots \sigma_{p_1} (U), \\ V_k &= \sigma_{\ell-p_k}^{-1} \sigma_{\ell-p_{k-1}}^{-1} \cdots \sigma_{\ell-p_1}^{-1} (V). \end{aligned}$$

By assumption  $U_k$  and  $V_k$  are double reverses, so there exist  $a_1, a_2, \dots, a_\ell \in F(X)$  such that

$$\begin{aligned} U_k &= (a_1, a_2, \dots, a_\ell), \\ V_k &= (a_\ell^*, a_{\ell-1}^*, \dots, a_1^*). \end{aligned}$$

Now let  $p = p_{k+1}$ ,  $U_{k+1} = \sigma_p(U_k)$ , and  $V_{k+1} = \sigma_{\ell-p}^{-1}(V_k)$ . The Hurwitz move  $\sigma_p$  only modifies elements at positions  $p$  and  $p + 1$  in  $U_k$ , and  $\sigma_{\ell-p}^{-1}$  only modifies elements at positions  $\ell - p$  and  $\ell - p + 1$  in  $V_k$ :

$$\begin{aligned} U_{k+1} &= (a_1, \dots, a_{p-1}, \underbrace{a_{p+1}}_p, \underbrace{a_{p+1}^{-1} a_p a_{p+1}}_{p+1}, a_{p+2}, \dots, a_\ell), \\ V_{k+1} &= (a_\ell^*, \dots, a_{p+2}^*, \underbrace{a_{p+1}^* a_p^* (a_{p+1}^*)^{-1}}_{\ell-p}, \underbrace{a_{p+1}^*}_{\ell-p+1}, a_{p-1}^*, \dots, a_1^*). \end{aligned}$$

Reversing the order of elements in  $U_{k+1}$  and changing each  $a_i$  to  $a_i^*$  produces a factorization identical to  $V_{k+1}$  except at position  $\ell - p$ , so the two factorizations are double reverses if factor  $\ell - p$  in  $V_{k+1}$  is the reverse of factor  $p + 1$  in  $U_{k+1}$ . But

$$(a_{p+1}^{-1} a_p a_{p+1})^* = a_{p+1}^* a_p^* (a_{p+1}^{-1})^* = a_{p+1}^* a_p^* (a_{p+1}^*)^{-1}$$

by (2) and Proposition 6.1, so  $U_{k+1}$  and  $V_{k+1}$  are indeed double reverses. □



**Theorem 6.6** Let  $G$  be a group given by a presentation  $\langle X \mid R \rangle$  with reversible relations. Let  $U$  be a length- $\ell$  factorization of elements in  $G$ , and let  $V$  be a double reverse of  $U$ . Then  $U$  and  $V$  have Hurwitz orbits of the same size.

**Proof.** Let  $O_U$  and  $O_V$  be the orbits of  $U$  and  $V$ , respectively. Suppose  $T \in O_U$  and  $T = \sigma_{p_1} \sigma_{p_2} \dots \sigma_{p_n}(U)$  with each  $p_i \in \{1, 2, \dots, \ell - 1\}$ , and define  $\varphi : O_U \rightarrow O_V$  by  $\varphi(T) = \sigma_{\ell-p_1}^{-1} \sigma_{\ell-p_2}^{-1} \dots \sigma_{\ell-p_n}^{-1}(V)$ . By Lemma 6.5,  $T$  and  $\varphi(T)$  are double reverses.

Assume  $A, B \in O_U$ ,  $C \in O_V$ , and  $\varphi(A) = \varphi(B) = C$ . Then there exist  $a_1, a_2, \dots, a_\ell, b_1, b_2, \dots, b_\ell \in F_n$  such that

$$A = (a_1, a_2, \dots, a_\ell) \quad \text{and} \quad B = (b_1, b_2, \dots, b_\ell),$$

$$C = (a_\ell^*, a_{\ell-1}^*, \dots, a_1^*) = (b_\ell^*, b_{\ell-1}^*, \dots, b_1^*).$$

Since  $a_i^* \equiv b_i^*$ , by Proposition 6.3  $a_i \equiv b_i$ ,  $i = 1, \dots, \ell$ . Thus  $A = B$ ,  $\varphi$  is injective, and the same process gives an injective map  $\varphi' : O_V \rightarrow O_U$  as well. This proves  $|O_U| = |O_V|$ .  $\square$

**Corollary 6.7** Suppose  $\langle X \mid R \rangle$  is a presentation for  $G$  with reversible relations, and let  $x_{i_1}, x_{i_2}, \dots, x_{i_\ell} \in X \cup X^{-1}$ . Then  $(x_{i_1}, x_{i_2}, \dots, x_{i_\ell})$  and  $(x_{i_\ell}, x_{i_{\ell-1}}, \dots, x_{i_1})$  have Hurwitz orbits of the same size.

**Proof.** The two factorizations are double reverses because the reverse of a single letter word is itself.  $\square$

## 7 Applications to reflection groups

A lot of the research done on the Hurwitz action has been on real reflection groups, and more generally on Coxeter groups. The reflections in these groups exclusively have order 2. Therefore, by Corollary 5.2, we find that a reflection factorization  $(x_1, x_2, \dots, x_n)$  in a Coxeter group has the same sized Hurwitz orbit as  $(x_n, \dots, x_2, x_1)$ .

The Hurwitz action has also been studied in the context of complex reflection groups called *Shephard groups*. Unlike Coxeter groups, Shephard groups can have generating reflections of order greater than 2. Thus Corollary 5.2 is not enough to guarantee that reversing a factorization of generators preserves Hurwitz orbit size. For each Shephard group there exist integers  $p_1, p_2, \dots, p_n$  and  $q_1, q_2, \dots, q_{n-1}$  such that there is a generating set  $\{s_1, s_2, \dots, s_n\}$  with relations  $s_i^{p_i} = 1$  for  $i = 1, 2, \dots, n$ ,  $s_i s_j = s_j s_i$  if  $|i - j| > 1$ , and

$$s_i s_{i+1} s_i s_{i+1} \dots = s_{i+1} s_i s_{i+1} s_i \dots \quad (q_i \text{ terms on each side})$$

for  $i = 1, 2, \dots, n - 1$ . We see with Remark 6.2 that these relations are reversible, so by Corollary 6.7 reversing any factorization consisting of generators  $s_i$  in a Shephard group preserves Hurwitz orbit size. For example, the group  $G_6$  mentioned in the introduction is a Shephard group, with parameters  $n = 2$ ,  $p_1 = 3$ ,  $p_2 = 2$ , and  $q_1 = 6$ . Yet testing of small cases has led us to make a stronger conjecture about Hurwitz orbit sizes in  $G_6$ .



**Conjecture 7.1** Let  $G_6 = \langle a, b \mid a^3 = b^2 = 1, ababab = bababa \rangle$ , and suppose  $x_1, x_2, \dots, x_\ell$  are elements of  $\{a, b, a^{-1}\}$  for some positive integer  $\ell$ . Then for any permutation  $\pi$  of  $\{1, 2, \dots, \ell\}$ ,  $(x_1, x_2, \dots, x_\ell)$  and  $(x_{\pi(1)}, x_{\pi(2)}, \dots, x_{\pi(\ell)})$  have Hurwitz orbits of equal size.

With regard to this property,  $G_6$  may be unique: We found that for each exceptional irreducible complex reflection group generated by 2 reflections  $a$  and  $b$ , excluding  $G_6$ , the factorizations  $(a, b, a, b)$  and  $(a, a, b, b)$  have Hurwitz orbits of different size.

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## References

- [1] B. Baumeister, M. Dyer, C. Stump, P. Wegener, A note on the transitive Hurwitz action on decompositions of parabolic Coxeter elements, *Proc. Amer. Math. Soc. Ser. B*, **1** (2014), 149–154.
- [2] T. Ben-Itzhak, M. Teicher, Graph theoretic method for determining Hurwitz equivalence in the symmetric group, *Israel J. Math.*, **135** (2003), 83–91.
- [3] E. Berger, Hurwitz equivalence in dihedral groups, *Electron. J. Combin.*, **18** (2011), #P45.
- [4] D. Bessis, Finite complex reflection arrangements are  $K(\pi, 1)$ , *Ann. of Math.*, **181** (2015), 809–904.
- [5] G. Gawankar, D. Lazreq, M. Rai, S. Sabar, Hurwitz actions on reflection factorizations in complex reflection group  $G_6^n$ , arXiv:2002.05102 (2020).
- [6] X. Hou, Hurwitz equivalence in tuples of generalized quaternion groups and dihedral groups, *Electron. J. Combin.*, **15** (2008), #R80.
- [7] A. Hurwitz, Ueber Riemann'sche Flächen mit gegebenen Verzweigungspunkten, *Math. Ann.*, **39** (1891), 1–60.
- [8] K. Igusa, R. Schiffler, Exceptional sequences and clusters, *J. Algebra*, **323** (2010), 2183–2202.
- [9] P. Kluitmann, Hurwitz action and finite quotients of braid groups, *Contemp. Math.*, **78** (1988), 299–325.
- [10] J.B. Lewis, V. Reiner, Circuits and Hurwitz action in finite root systems, *New York J. Math.*, **22** (2016), 1457–1486.
- [11] J.B. Lewis, A note on the Hurwitz action on reflection factorizations of Coxeter elements in complex reflection groups, *Electron. J. Combin.*, **27** (2020), P2.54.
- [12] H. Mühle, V. Ripoll, Connectivity properties of factorization posets in generated groups, *Order*, **37** (2020), 115–149.
- [13] D. Peifer, An introduction to combinatorial group theory and the word problem, *Math. Mag.*, **70** (1997), 3–10.
- [14] Z. Peterson, Hurwitz transitivity of longer reflection factorizations in  $G_4$  and  $G_5$ , arXiv:1808.01268 (2018).



- [15] C. Sia, Hurwitz equivalence in tuples of dihedral groups, dicyclic groups, and semidihedral groups, *Electron. J. Combin.*, **16** (2009), #R95.

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