Watts Spaces and Smooth Maps

B. Virgin

Abstract - A new category is described, which generalizes a select variety of categories having smooth mappings as their class of morphisms, those like the C^∞ manifolds and C^∞ mappings between them. Categorical embeddings are produced to justify this claim of generalization. Theorems concerning the equivalence of smooth and continuous versions of different separation axioms are proved following the categorical discussion. These are followed by a generalization of Whitney's approximation theorem, a smooth version of the Tietze extension theorem, and a sufficient condition to guarantee that the connected components of these spaces are smoothly path-connected.

Keywords : Souriau (or diffeological) space; Sikorski (or differential) space; Frölicher space; smoothness; separation; approximation; extension

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1 Introduction

The subject of differential topology began with the study of finite dimensional smooth manifolds, objects which retained many of the features of the euclidean spaces studied classically. These incredibly specialized spaces have more than enough structure to allow for the smoothness of maps between them to be defined, and for many important theorems in elementary calculus to be extended in a natural way. They also possess enough structure to ensure many other desirable properties, like the equivalence of all notion of tangent space, connectedness implying smooth path connectedness, and continuous homotopy implying smooth homotopy for smooth maps. These spaces are also incredibly relevant to the physical sciences, being central not only to General Relativity, but also to the modern gauge-symmetry approach to Quantum Field Theory.

Ever since the inception of smooth manifolds, it was understood that there were larger categories at work, as manifolds with boundaries were quickly introduced alongside the original spaces, followed shortly thereafter by those with a boundary and corners. In those places where the definition of smooth manifold was found to be insufficient, many authors of a variety of motivations produced new classes of objects which could still allow for a consistent notion of smoothness for the mappings between them. At present, the most general conceptions of smoothness defined using only topological terminology can be found in the categories of Frölicher, Sikorski, and Souriau spaces. The category of Frölicher spaces can be embedded as full subcategories of the categories of Sikorski spaces (often called differential spaces) and Souriau spaces (often called diffeological spaces)

[1, 7], while the category of smooth manifolds can itself be embedded in the category of Frölicher spaces as a full subcategory. This means that the smooth manifolds can be embedded as full subcategories in all three of these new categories, making each of them a generalization of the smooth manifolds.

Souriau and Sikorski spaces in particular have the claim of forming the most general categories in this presented lineup, but their conceptions of smoothness are not coincident. This naturally raises the question of whether or not there is some larger category whose class of morphisms can still justifiably be referred to as smooth maps. Jordan Watts, in his thesis [7], suggested the class of objects for such a category, sets bestowed with both a structure of Souriau and of Sikorski's definitions. He added that in addition to this pair of structures, a meaningful new space would require that these two structures are compatible with one another, in the sense that compositions formed from this pair should belong to the class of smooth maps between open subsets of euclidean spaces.

In this paper, Watts' category will be completed with a definition for smooth mapping between his spaces, and it will be shown that the Souriau spaces and Sikorski spaces, and by extension Frölicher spaces and smooth manifolds, may be embedded as full subcategories into Watts'. This will be followed with a demonstration that several of the beloved topological qualities of smooth manifolds are also found in Watts spaces, such as the equivalence of several notions of separation, the prospect of extension of smooth mappings, and approximation of continuous mappings by smooth ones. Finally, a condition is produced, which when required of a Watts space guarantees smooth path-connectedness of all connected components.

2 The Category of Watts Spaces

Set will be used to denote the category of sets and functions, while C^∞ wille be short for the category of smooth manifolds and C^∞ mappings, where here C^∞ means that every local representation of a function is infinitely continuously differentiable. For any category **Cat**, and for any objects A and B in **Cat**'s class of objects the corresponding class of morphisms going from A to B will be denoted $\mathbf{Cat}(A, B)$.

Definition 2.1 *A **Souriau Space** is an ordered pair (X, P) where X is a set, and*

$$P \subset \bigcup_{n=0}^{\infty} \left(\bigcup_{\substack{U \text{ is open} \\ \text{in } \mathbb{R}^n}} \boldsymbol{Set}(U, X) \right),$$

*which satisfies the **Souriau Axioms**:*

1) P contains every constant map in the set

$$\bigcup_{n=0}^{\infty} \left(\bigcup_{\substack{U \text{ is open} \\ \text{in } \mathbb{R}^n}} \boldsymbol{Set}(U, X) \right).$$

2) If $p \in P$, and

$$f \in \bigcup_{n=0}^{\infty} \left(\bigcup_{\substack{U \text{ is open} \\ \text{in } \mathbb{R}^n}} C^{\infty}(U, Dom(p)) \right),$$

then $p \circ f \in P$.

3) If there exists a function

$$f \in \bigcup_{n=0}^{\infty} \left(\bigcup_{\substack{U \text{ is open} \\ \text{in } \mathbb{R}^n}} \boldsymbol{Set}(U, X) \right),$$

such that there also exists an accompanying open cover $\{U_j\}_{j \in J}$ of $Dom(f)$, and a set of functions $\{p_j\}_{j \in J} \subset P$ for which $f|_{U_j} = p_j|_{U_j}$ holds true for every $j \in J$, then $f \in P$.

A set P satisfying these axioms is called a ***Souriau Structure****, and members of P will be referred to as* ***Plots****.*

Smoothness for mappings between Souriau spaces is then defined as the preservation of Souriau structures under composition.

Definition 2.2 *Let (X, P_X) and (Y, P_Y) be Souriau spaces, a function $f : X \to Y$ is* ***Souriau Smooth*** *if $f \circ p \in P_Y$ for every $p \in P_X$.*

Together these objects and maps form a category, abbreviated **Sou**, which is investigated thoroughly in [3].

Definition 2.3 *A* ***Sikorski Space*** *is an ordered pair (X, C) where X is a set, and $C \subset \boldsymbol{Set}(X, \mathbb{R})$ satisfying the* ***Sikorski Axioms****:*

1) C contains all possible constant maps in $\boldsymbol{Set}(X, \mathbb{R})$.

2) if $f \in C^{\infty}(\mathbb{R}^n, \mathbb{R})$, and $\{c_j\}_{j=1}^{n} \subset C$, then $f(c_1, c_2, ..., c_n) \in C$.

3) if $f \in \boldsymbol{Set}(X, \mathbb{R})$ such that there is an open cover $\{U_j\}_{j \in J}$ of X (in the initial topology determined by C) and a set of functions $\{c_j\}_{j \in J} \subset C$ such that $f|_{U_j} = c_j|_{U_j}$ for every $j \in J$, then $f \in C$.

A set of maps C obeying these axioms is called a ***Sikorski Structure****, and the elements it contains will be called* ***Coplots****.*

This definition is modified slightly from the standard presented in the beginning of [6], to more clearly illustrate the parallels between the Souriau axioms and the Sikorski axioms. Generally the Sikorski structure is also assumed to be nonempty, which turns the first axiom into a theorem depending on the second axiom and non-emptiness. Sikorski spaces have an accompanying notion of smoothness parallel to that defined for Souriau spaces.

Definition 2.4 *Let (X, C_X) and (Y, C_Y) be Sikorski spaces, a function $f : X \to Y$ is* ***Sikorski smooth*** *if $c \circ f \in C_X$ for all $c \in C_Y$.*

Sikorski spaces together with Sikorski smooth maps form a category, abbreviated here as **Sik**. These Sikorski and Souriau structures, as they are presented here, have axioms which appear to be quite dual to one another. Moreover, these axioms represent the minimal requirements necessary for a definition of smoothness.

The axiom regarding the inclusion of constant maps is necessary because smoothness generalizes infinite continuous differentiability, and constant maps are the most continuously differentiable variety of functions. The axiom allowing for smooth compositions of maps on the euclidean side of our plots and coplots is necessary if we are to consider the elements of these structures to be smooth maps, as smoothness ought to be a preserved under compositions, and doing so would ensure that we could eventually form a category with these maps as the morphisms. Finally, the last axiom found in either definition is responsible for guaranteeing that the new notion of smoothness is still a local property, in the same way that continuity is a local property.

Definition 2.5 *A **Watts Space** is an ordered triple (X, P, C) where X is a set, P is a Souriau structure, C is a Sikorski structure, and the pair (P, C) satisfies **Watts' Axiom***

$$(c \circ p) \in \bigcup_{n=0}^{\infty} \left(\bigcup_{\substack{U \text{ is open} \\ \text{in } \mathbb{R}^n}} C^{\infty}(U, \mathbb{R}) \right) \text{ for every } (c, p) \in C \times P. \tag{1}$$

*A pair (P, C) satisfying these requirements will be referred to as a **Watts Structure**.*

The minimality of required axioms in this object class is preserved, as Watts' suggested axiom is an extension of the second Souriau and Sikorski axioms, in that it simply continues to enforce the preservation of smoothness under compositions. Any set can be made into a Watts space, as this category admits a trivial object for every element of **Set**'s object class.

Definition 2.6 *Given a set X, the **Trivial Structure** associated to it is the pair (P, C), where:*

$$P = \left\{ p \in \bigcup_{n=0}^{\infty} \left(\bigcup_{\substack{U \text{ is open} \\ \text{in } \mathbb{R}^n}} \textbf{\textit{Set}}(U, X) \right) \middle| p \text{ is constant} \right\}, \text{ and}$$

$$C = \{c \in \textbf{\textit{Set}}(X, \mathbb{R}) | c \text{ is constant}\}.$$

More interestingly, all finite dimensional smooth manifolds can be interpreted as Watts spaces in a natural way.

Definition 2.7 *Given a finite-dimensional smooth manifold M, the **Standard Structure** associated to it is the pair (P, C), where:*

$$P = \bigcup_{n=0}^{\infty} \left(\bigcup_{\substack{U \text{ is open} \\ \text{in } \mathbb{R}^n}} C^{\infty}(U, M) \right), \text{ and}$$

$$C = C^{\infty}(M, \mathbb{R}).$$

Any smooth manifold, such as one of the euclidean spaces, will always be assumed to be equipped with this standard structure unless otherwise specified. Much like the smooth manifolds, where one begins with a topological manifold and bestows it with a smooth structure, we have begun with a set, and equipped it with sets of maps which constitute a smooth structure. There are notable differences between the two approaches, first of which being that in the case of Watts spaces, the topology is determined by the structure, as we will see, but a smooth manifold's topology was implicit to the base object before it was made smooth. A second difference is that a Watts structure does not mandate that the Watts space it is defining is locally euclidean throughout, and so the diffeomorphism groups of the connected components of a Watts space need not act transitively on their respective components.

Definition 2.8 *Let (X, P_X, C_X) and (Y, P_Y, C_Y) be Watts spaces, and let $A \subset X$ be an arbitrary subset. A map $f : A \to Y$ will be called **Smooth** if:*

1) $A = X$, f is both Souriau smooth for the Souriau spaces (X, P_X) and (Y, P_Y) and Sikorski smooth for the Sikorski spaces (X, C_X) and (Y, C_Y)

2) $A \neq X$, there is a map $f^* : X \to Y$ which is smooth according to the first case, such that $f^*|_A = f$.

This notion of smoothness just entails the combined requirements from the definitions of smoothness in the Souriau and Sikorski categories. The restriction case has been added to this definition in order to allow for certain theorems to be stated while avoiding the discussion of sub-object constructions, which will not be defined here.

By bestowing a set with a Watts structure, one has declared those members of the Watts structure to be exactly the set of smooth maps between euclidean spaces and the set in question, this declaration then enforces the new notion of smoothness on all maps between these spaces.

Theorem 2.9 *If (X, P, C) is a Watts space then a map*

$$p \in \bigcup_{n=0}^{\infty} \left(\bigcup_{\substack{U \text{ is open} \\ \text{in } \mathbb{R}^n}} \boldsymbol{Set}(U, X) \right),$$

is smooth if and only if $p \in P$, and similarly, a map $c \in \boldsymbol{Set}(X, \mathbb{R})$ is smooth if and only if $c \in C$.

Proof. Let (X, P, C) be a Watts space, and let $p \in P$, then for any plot in the standard Watts structure on $\mathrm{Dom}(p)$:

$$f \in \bigcup_{n=0}^{\infty} \left(\bigcup_{\substack{U \text{ is open} \\ \text{in } \mathbb{R}^n}} C^{\infty}(U, \mathrm{Dom}(p)) \right),$$

the second Souriau axiom allows us to conclude that $p \circ f \in P$. For any $c \in C$, $c \circ p \in \mathbf{Set}(\mathrm{Dom}(p), \mathbb{R})$, while the Watts axiom guarantees that

$$c \circ p \in C^\infty(\mathrm{Dom}(p), \mathbb{R}).$$

Therefore $c \circ p \in C^\infty(\mathrm{Dom}(p), \mathbb{R})$, and so it is a coplot in $\mathrm{Dom}(p)$'s standard structure. Thus every $p \in P$ is indeed smooth.

Now assume that a map

$$p \in \bigcup_{n=0}^{\infty} \left(\bigcup_{\substack{U \text{ is open} \\ \text{in } \mathbb{R}^n}} \mathbf{Set}(U, X) \right),$$

is smooth. Since $\mathrm{Id} : \mathrm{Dom}(p) \to \mathrm{Dom}(p)$ is C^∞, so it is a plot on $\mathrm{Dom}(p)$, which means that $p = p \circ \mathrm{Id} \in P$ by the definition of smooth map. Therefore a map $p \in \bigcup_{n=0}^{\infty} \mathbf{Set}(\mathbb{R}^n, X)$ can only be smooth if it is a plot.

Let $c \in C$, for any coplot $f \in C^\infty(\mathbb{R}, \mathbb{R})$ in $\mathbb{R}$'s standard structure, the second Sikorski axiom guarantees that $f \circ c \in C$. For any $p \in P$, the Watts axiom guarantees that

$$c \circ p \in \bigcup_{n=0}^{\infty} \left(\bigcup_{\substack{U \text{ is open} \\ \text{in } \mathbb{R}^n}} C^\infty(U, \mathbb{R}) \right).$$

From this we can deduce that $c \circ p$ is one of the plots in $\mathbb{R}$'s standard structure, and this allows us to conclude that all coplots $c \in C$ are smooth.

Now assume that $c \in \mathbf{Set}(X, \mathbb{R})$ is a smooth map, then since $\mathrm{Id} : \mathbb{R} \to \mathbb{R}$ is C^∞ it is a coplot in $\mathbb{R}$'s standard structure. This together with the definition of smooth map imply that $c = \mathrm{Id} \circ c \in C$. Therefore coplots are exactly the smooth real valued functions on a Watts space. □

To see that this definition of smoothness is well-chosen from the categorical point of view, a demonstration of the compatibility of this definition with ordinary function composition is required.

Theorem 2.10 *Compositions of smooth maps are smooth.*

Proof. Let $f : A \to Y$ and $g : B \to Z$ be smooth maps for Watts spaces (X, P_X, C_X), (Y, P_Y, C_Y), and (Z, P_Z, C_Z) where $A \subset X$, $B \subset Y$, and $f(A) \subset B$. By the definition of smoothness, there must be smooth maps $f^* : X \to Y$ and $g^* : Y \to Z$ such that $f^*|_A = f$ and $g^*|_B = g$.

For any $p \in P_X$, $f^* \circ p \in P_Y$ by f^*'s smoothness, and so $g^* \circ (f^* \circ p) = (g^* \circ f^*) \circ p \in P_Z$ by the smoothness of g^*. For any $c \in P_Z$, $c \circ g^* \in C_Y$ by g^*'s smoothness, and so $c \circ (g^* \circ f^*) = (c \circ g^*) \circ f^* \in C_X$ by the smoothness of f^*. Therefore $g^* \circ f^* : X \to Z$ is a smooth map.

Since $f^*|_A = f$, and so $f^*(A) = f(A) \subset B$, we must also have that $(g^* \circ f^*)|_A = g^*|_B \circ f^*|_A = g \circ f$. Therefore $g \circ f$ is smooth, and so we are justified in concluding that smoothness is preserved by standard function composition. □

The last piece of confirmation necessary to indicate that this choice of definition suffices to produce a category is that it is flexible enough to be inclusive of all identity maps, which will leave this class of morphisms with enough structure to form a category when paired with the class of all Watts spaces.

Theorem 2.11 *The Identity map of any Watts space is a smooth map.*

Proof. Let (X, P, C) be a Watts space and $\mathrm{Id} : X \to X$ be the identity map on X. Then for any $c \in C$, $c \circ \mathrm{Id} = c \in C$, and for any $p \in P$, $\mathrm{Id} \circ p = p \in P$, therefore Id is smooth by definition. □

Together these results show us that the objects and mappings defined here possess the requisite structure to produce a category.

Theorem 2.12 *The class of all Watts spaces, the class of all smooth maps between Watts spaces, and ordinary function composition together form a category.*

Proof. That our composition rule is associative follows from the associativity of ordinary function composition, while the closure of the class of smooth maps between Watts spaces under composition of its members is justified by theorem 2.10. That members of the class of all Watts spaces have identity morphisms in the class of all smooth maps between Watts spaces follows immediately from proposition 2.11, that all identity maps are smooth, therefore the pair of collections described does indeed form a category. □

Now that we are completely justified in referring to it as a category, the class of Watts spaces together with the class of all smooth maps will be abbreviated as the category **Wat**. With access to the category described by Watts now firmly established, the next order of business is to prove Watts' claim that a category with spaces fitting his description should generalize the previous categories bearing smooth maps. It won't be necessary to show this for all four such categories previously referenced, as those of Sikorski and Souriau spaces both generalize the categories of Frölicher spaces and smooth manifolds. Therefore a demonstration that the Sikorski and Souriau categories can be embedded as full subcategories of **Wat** will be sufficient to prove this fact for all relevant cases.

Theorem 2.13 *There exists categorical embeddings from **Sik** and **Sou** to **Wat**.*

Proof. Define the functors $F : \mathbf{Sik} \to \mathbf{Wat}$ and $G : \mathbf{Sou} \to \mathbf{Wat}$ in the following way:

$$F(X, C) = \left(X, \left\{ p \in \bigcup_{n=0}^{\infty} \mathbf{Set}(\mathbb{R}^n, X) | c \circ p \text{ is smooth } \forall c \in C \right\}, C \right),$$

$$F(f : X_1 \to X_2) = f : X_1 \to X_2,$$

$$G(Y, P) = (Y, P, \{c \in \mathbf{Set}(Y, \mathbb{R}) | c \circ p \text{ is smooth } \forall p \in P\}),$$

$$G(g : Y_1 \to Y_2) = g : Y_1 \to Y_2.$$

That $F(X, C)$ and $G(Y, P)$ are indeed Watts spaces follows from proposition 2.7 of [1], where it is shown that the procedure of selecting all maps that could possibly compose smoothly with the starting structure always generates the opposing structure beginning from a Sikoski structure or a Souriau structure, and it is clear the image pairs must satisfy Watts' axiom from their definition.

Let $f : X_1 \to X_2$ be Sikorski smooth for Sikorski spaces (X_1, C_1) and (X_2, C_2). $F(f) = f$ will send coplots to coplots by the definition of Sikorski smooth, so let p belong to $F(X_1, C_1)$'s set of plots. For any $c \in C_2$, $c \circ f \in C_1$ by Sikorski smoothness, therefore $(c \circ f) \circ p = c \circ (f \circ p)$ is a smooth map by the Watts axiom. Since this is true for every $c \in C_2$, and $F(X_2, C_2)$'s set of plots contains every map that composes smoothly with all members of C_2, $f \circ p$ must be one of $F(X_2, C_2)$'s plots, and so $F(f)$ is smooth.

Let $g : Y_1 \to Y_2$ be Souriau smooth for Souriau spaces (Y_1, P_1) and (Y_2, P_2). $F(g) = g$ will send plots to plots by the definition of Souriau smooth, so let c belong to $G(Y_2, P_2)$'s set of coplots. For any $p \in P_1$, $g \circ p \in P_2$ by Souriau smoothness, therefore $c \circ (g \circ p) = (c \circ g) \circ p$ is a smooth map by the Watts axiom. Since this is true for every $p \in P_1$, and $G(Y_1, P_1)$'s set of coplots contains every map that composes smoothly with all members of P_1, $c \circ g$ must be one of $G(Y_1, P_1)$'s coplots, and so $G(g)$ is smooth.

That these definitions preserve identities and compositions is true trivially due to the fact that these functors are exactly the identity mappings on their classes of morphisms. Therefore these functors are at the very least well-defined.

The faithfulness of these functors is also a trivial consequence of having both been defined as the identity on their classes of morphisms. Fullness as well, since the definition of smoothness presented here is stronger than either Sikorski smoothness or Souriau smoothness (being the combination of the two), any Watts smooth map is itself both a Sikorski smooth map and Souriau smooth map, meaning this map is in the domains of G and F so that it can always be mapped to itself.

To see that these functors are indeed injective on objects, note that if $F(X_1, C_1) = F(X_2, C_2)$ then F's definition immediately implies that $X_1 = X_2$ and $C_1 = C_2$ since those components correspond, and so $(X_1, C_1) = (X_2, C_2)$. If, on the other hand, $G(Y_1, P_1) = G(Y_2, P_2)$, then G's definition implies that $Y_1 = Y_2$ and $P_1 = P_2$, leaving us with $(Y_1, P_1) = (Y_2, P_2)$. Therefore our functors $F : \mathbf{Sik} \to \mathbf{Wat}$ and $G : \mathbf{Sou} \to \mathbf{Wat}$ are indeed fully faithful, and injective on objects, and so they are indeed categorical embeddings. $\square$

As was previously mentioned, there are similar embeddings of the category of smooth manifolds into the category of Frölicher spaces, and of the category of Frölicher spaces into the categories of Souriau spaces and Sikorski spaces [1, 7]. This means that all four categories can be fully embedded into **Wat**, and so this category naturally extends all of these prior smooth categories. Like **Sik** and **Sou**, **Wat** is very amenable to constructions, for instance the generation of structures from pairs of generating sets.

Definition 2.14 *Given a set X, and a pair of sets*

$$P^* \subset \bigcup_{n=0}^{\infty} \left(\bigcup_{\substack{U \text{ is open} \\ \text{in } \mathbb{R}^n}} \boldsymbol{Set}(U, X) \right) \text{ and } C^* \subset \boldsymbol{Set}(X, \mathbb{R}),$$

for which

$$c^* \circ p^* \in \bigcup_{n=0}^{\infty} \left(\bigcup_{\substack{U \text{ is open} \\ \text{in } \mathbb{R}^n}} C^\infty(U, \mathbb{R}) \right) \text{ for every } (c,p) \in C \times P,$$

*the **Generated Structure** is the pair here denoted (P, C).*

P is defined to be the set of all maps

$$p \in \bigcup_{n=0}^{\infty} \left(\bigcup_{\substack{U \text{ is open} \\ \text{in } \mathbb{R}^n}} \boldsymbol{Set}(U, X) \right),$$

for which there exists an open cover $\{U_j\}_{j\in J}$ of $Dom(p)$ such that for all $j \in J$ the restriction $p|_{U_j}$ is a constant map or equal to a restriction $(p^* \circ f)|_{U_j}$ for some $f \in C^\infty(Dom(p), Dom(p^*))$ and a $p^* \in P^*$.

C consists of all maps $c \in \boldsymbol{Set}(X, \mathbb{R})$ for which there exists an open cover $\{U_j\}_{j\in J}$ of X in the initial topology determined by C^* such that each restriction $c|_{U_j}$ is constant or equal to a restriction $f(c_1, c_2, ..., c_n)|_{U_j}$ for some $f \in C^\infty(\mathbb{R}^n, \mathbb{R})$ and a finite subset of the coplot generator $\{c_l\}_{l=1}^n \subset C^*$.

As the name would suggest, the structure generated from a pair such as the one found above is indeed a Watts structure. Moreover, preservation of the components of these generating pairs under composition is a necessary and sufficient condition to imply the smoothness of a mapping, as can be seen from the following proposition.

Theorem 2.15 *Generated Watts structures are indeed Watts structures. Additionally, if two sets X and Y are each given the generating pairs (P_X^*, C_X^*) and (P_Y^*, C_Y^*) respectively, $A \subset X$ and $f : A \to Y$ is a map with an extension $f^* : X \to Y$ for which $c^* \circ f^* \in C_X^*$, and so that $f^* \circ p^* \in P_Y^*$ for all $p^* \in P_X^*$ and $c^* \in C_Y^*$, then f is smooth for the generated Watts spaces.*

Proof. Given a set X, and a pair of sets:

$$P^* \subset \bigcup_{n=0}^{\infty} \left(\bigcup_{\substack{U \text{ is open} \\ \text{in } \mathbb{R}^n}} \mathbf{Set}(U, X) \right) \text{ and } C^* \subset \mathbf{Set}(X, \mathbb{R}),$$

for which

$$c^* \circ p^* \in \bigcup_{n=0}^{\infty} \left(\bigcup_{\substack{U \text{ is open} \\ \text{in } \mathbb{R}^n}} C^\infty(U, \mathbb{R}) \right) \text{ for every } (c,p) \in C \times P,$$

then P as defined above is indeed a Souriau structure according to lemma 1.12 in [4]. Similarly, given C^* the set C as expressed in the same definition is a Sikorski structure according to proposition 2.26 of [7].

To see that the pair (P, C) must be a Watts structure, let $c \in C$ and $p \in P$. This means that there exists an open cover $\{U_j\}_{j\in J}$ of X in the initial topology determined by C^* such that each restriction $c|_{U_j}$ is constant or equal to $f(c_1^*, c_2^*, ..., c_n^*)|_{U_j}$ for some $f \in C^\infty(\mathbb{R}^n, \mathbb{R})$ and a finite subset $\{c_l^*\}_{l=1}^n \subset C^*$. It also means that there exists an open cover $\{Q_k\}_{k\in K}$ of $\mathrm{Dom}(p)$ such that each $p|_{Q_k}$ is constant or equal to $p^* \circ g|_{Q_k}$ where g is a C^∞ map between euclidean spaces and $p^* \in P^*$.

Note that for any $c^* \in C^*$, $c^* \circ p$ must be C^∞ because there is an open cover $\{Q_k\}_{k\in K}$ of its domain $\mathrm{Dom}(p)$ such that $(c^* \circ p)|_{Q_k} = c^* \circ (p|_{Q_k})$ is constant or equal to $c^* \circ (p^* \circ g)|_{Q_k} = (c^* \circ p^* \circ g)|_{Q_k}$ where g is a C^∞ map between euclidean spaces and $p^* \in P^*$.

$\{p^{-1}(U_j)\}_{j\in J}$ is also an open cover of $\mathrm{Dom}(p)$, since $\{U_j\}_{j\in J}$ is part of the initial topology determined by C^* and $c^* \circ p$ is continuous for every $c^* \in C^*$. Therefore

$$\{V_{j,k} = p^{-1}(U_j) \cap Q_k\}_{(j,k)\in J\times K}, \tag{2}$$

is also an open cover of $\mathrm{Dom}(p)$. Moverover, since $V_{j,k} \subset Q_k$, the open cover $\{V_{j,k}\}_{(j,k)\in J\times K}$ has been chosen such that $p|_{V_{j,k}} = (p|_{Q_k})|_{V_{j,k}}$ must be constant or equal to $(p^* \circ g)|_{V_{j,k}} = ((p^* \circ g)|_{Q_k})|_{V_{j,k}}$ where g is a C^∞ map between euclidean spaces and $p^* \in P^*$.

On the other hand, $V_{j,k} \subset p^{-1}(U_j)$, so that $p(V_{j,k}) = \mathrm{Ran}(p|_{V_{j,k}}) \subset U_j$, meaning that

$$c \circ (p|_{V_{j,k}}) = (c|_{U_j}) \circ (p|_{V_{j,k}}) = f(c_1^*, c_2^*, ..., c_n^*)|_{U_j} \circ (p^* \circ g)|_{V_{j,k}}$$

$$= f(c_1^* \circ p^* \circ g, c_2^* \circ p^* \circ g, ..., c_n^* \circ p^* \circ g)|_{V_{j,k}},$$

or else $(c \circ p)|_{V_{j,k}} = c \circ (p|_{V_{j,k}})$ is constant. If the restriction is equal to the composition shown above, then it must be C^∞ because f, g, and all of the compositions $c_l^* \circ p^*$ for all $(c_l^*, p^*) \in C^* \times P^*$ are C^∞, and it must certainly be C^∞ if it is constant. Therefore there is an open cover of $\mathrm{Dom}(p)$ such that $c \circ p$'s restrictions are all C^∞, therefore it is a C^∞ map from an open subset of a euclidean space to the real numbers. From this we may conclude that the pair (P, C) satisfies the Watts axiom.

Let X and Y be sets, with sets of maps P_X^*, C_X^*, P_Y^*, and C_Y^*, for which $c^* \circ p^*$ is C^∞ for any $(c^*, p^*) \in P_X^* \times C_X^*, P_Y^* \times C_Y^*$. Additionally, let $A \subset X$ and $f : A \to Y$ be a map with an extension $f^* : X \to Y$ such that for any $p^* \in P_X^*$ and any $c^* \in C_Y^*$, $f^* \circ p^* \in P_Y^*$ and $c^* \circ f^* \in C_X^*$.

For any p in the Souriau structure P_X generated by P_X^*, there is an open cover $\{U_j\}_{j\in J}$ of $\mathrm{Dom}(p)$ such that $p|_{U_j}$ is constant or equal to the restriction of a composition $p^* \circ g$ where g is C^∞ and $p^* \in P_X^*$. Therefore $f^* \circ p$ must have the property that $(f^* \circ p)|_{U_j}$ is constant or equal to the restriction of $f^* \circ p^* \circ g$. Since $f^* \circ p^* \in P_Y^*$, g is C^∞, and $\{U_j\}_{j\in J}$ covers $f^* \circ p$'s domain, $f^* \circ p$ must be a member of the Souriau structure P_Y generated by P_Y^*.

Note that for any set $V \subset Y$ open in the initial topology determined by C_Y^*, and any point $y \in V$, there must be a finite collection of maps $\{q_\ell\}_{\ell=1}^m$ and an associated collection of open subsets of $\mathbb{R}$: $\{W_\ell\}_{\ell=1}^m$, such that:

$$y \in \bigcap_{\ell=1}^m q_\ell^{-1}(W_\ell) \subset V. \tag{3}$$

Applying the preimage with respect to f^* to this last containment yields:

$$f^{*-1}(y) \in f^{*-1}\left(\bigcap_{\ell=1}^{m} q_\ell^{-1}(W_\ell)\right) = \bigcap_{\ell=1}^{m}(q_\ell \circ f^*)^{-1}(W_\ell) \subset f^{*-1}(V). \tag{4}$$

Which since every $x \in f^{*-1}(V)$ is contained in a fiber $f^{*-1}(y)$ for some $y \in V$, and $q \circ f^* \in C_X^*$ for all $q \in C_Y^*$, means that $f^{*-1}(V)$ is open in the initial topology determined by C_X^*. Therefore f^* is continuous when X and Y are given the initial topologies from C_X^* and C_Y^*, respectively.

For any c in the Sikorski structure C_Y generated by C_Y^*, there is an open cover $\{U_j\}_{j\in J}$ of Y in the initial topology determined by C_Y^* such that every restriction $c|_{U_j}$ is constant or else equal to the restriction of a composition $g(c_1, c_2, ..., c_n)$ for a C^∞ function g and a finite subset $\{c_k\}_{k=1}^n \subset C_Y^*$. By the continuity of f, $\{f^{*-1}(U_j)\}_{j\in J}$ is an open cover of X in the initial topology determined by C_X^* such that $c \circ f^*|_{f^{*-1}(U_j)} = c|_{U_j} \circ f^*|_{f^{*-1}(U_j)}$ is either constant or equal to the restriction of $g(c_1 \circ f^*, c_2 \circ f^*, ..., c_n \circ f^*)$, which since $\{f \circ c_1, f \circ c_2, ..., f \circ c_n\} \subset C_X^*$ by hypothesis means that $c \circ f$ must belong to the Sikorski structure generated by C_X^*.

Therefore f is indeed smooth for the generated spaces, and this argument also suggests that smoothness need only be tested on generating sets, rather than entire Watts structures. □

Having the ability to generate structures, the notion of product structure will be defined so as to facilitate later proofs. It should be noted that the natural definition, the one cast in terms of a natural structure inherited from the set of canonical projection mappings, has not been used in favour of the more computationally explicit definition found below. These definitions still coincide, but the demonstration of this fact would amount to a non-sequitur in the present discussion, so this proof will be found in a sequel.

Definition 2.16 *Given an indexed collection of sets $\{X_j\}_{j\in J}$, the jth* ***Canonical Projection*** *associated to the product $\prod_{j\in J} X_j$ is the map $\pi_j : \prod_{\ell\in J} X_\ell \to X_j$ defined $\pi_j(x) = x(j) \equiv x_j$ for each $\left(x : J \to \bigcup_{j\in J} X_j\right) \in \prod_{\ell\in J} X_\ell$.*

Definition 2.17 *Given a set X, and an index set J, the associated* ***Diagonal Map*** *$\Delta : X \to \prod_{j\in J} X$ is defined $\Delta(x) = \tilde{x} \in \prod_{j\in J} X$, for all $x \in X$ where $\tilde{x}_j = x$ for all $j \in J$.*

Definition 2.18 *Given a set of Watts spaces $\{(X_j, P_j, C_j)\}_{j\in J}$, the* ***Product Structure*** *$(P_{\prod_j X_j}, C_{\prod_j X_j})$ defined on the cartesian product $\prod_{j\in J} X_j$ is the structure with plots $P_{\prod_j X_j}$ generated from the set:*

$$\left\{\left(\prod_{j\in J} p_j\right) \circ \Delta : U \to \prod_{j\in J} X_j \,\middle|\, p_j \in P_j, Dom(p_j) = U \text{ for each } j \in J\right\}$$

and with coplots $C_{\prod X_j}$ generated from the set

$$\bigcup_{j\in J}\{c \circ \pi_j | c \in C_j \text{ and } \pi_j \text{ is the } j\text{th canonical projection.}\}.$$

Given this definition, we achieve the natural desirata, that products of smooth maps become smooth maps between corresponding product structures.

Theorem 2.19 *Cartesian products of smooth maps between Watts spaces are smooth on the resulting product Watts spaces.*

Proof. Let $\{f_j : X_j \to Y_j\}_{j\in J}$ be a set of smooth maps for sets of Watts spaces $\{(X_j, P_j, C_j)\}_{j\in J}$ and $\{(Y_j, P_j^*, C_j^*)\}_{j\in J}$, then for any member $c \circ \pi_j^*$ of the generating set of $C_{\prod_j Y_j}$, it is true that

$$(c \circ \pi_j^*) \circ \prod_{j\in J} f_j = c \circ \left(\pi_j^* \circ \prod_{j\in J} f_j\right) = c \circ (f_j \circ \pi_j) = (c \circ f_j) \circ \pi_j. \tag{5}$$

The composition above must be a member of the generating set for $C_{\prod_j X_j}$ because $c \circ f_j \in C_j$ on account of f_j being smooth and $c \in C_j^*$ for each $j \in J$.

For any $\prod_{j\in J} p_j$ with $\mathrm{Dom}(p_j) = U$ for all $j \in J$, we can see that

$$\left(\prod_{j\in J} f_j\right) \circ \left(\left(\prod_{j\in J} p_j\right) \circ \Delta\right) = \left(\prod_{j\in J} (f_j \circ p_j)\right) \circ \Delta. \tag{6}$$

Therefore the composition above must belong to the generating set of $\prod_{j\in J} Y_j$'s plots, because $(f_j \circ p_j) \in P_j^*$ since f_j is smooth and $p_j \in P_j$ for each $j \in J$. Since $\prod_{j\in J} f_j$ sends members of the various generating sets to the opposing generating set via composition, it must be smooth by theorem 2.15. □

Similarly, we can also obtain the result that all diagonal maps are smooth, yet another result displaying the close analogy of the category of Watts spaces and smooth maps to the category of topological spaces and continuous maps.

Theorem 2.20 *The diagonal map is smooth for any cartesian product equipped with the product structure.*

Proof. Let (X, P, C) be a Watts space, and Δ be the diagonal map, that is the unique map $\Delta : X \to \prod_{j\in J} X$ for which $(\pi_j \circ \Delta) = \mathrm{Id}$ for the canonical projections $\pi_j : \prod_{j\in J} X \to X$ and each $j \in J$.

For any plot $p \in P$, the composition $\Delta \circ p = \left(\prod_{j\in J} p\right) \circ \Delta^*$, where $\Delta^* : \mathrm{Dom}(p) \to \prod_{j\in J} \mathrm{Dom}(p)$ is the diagonal map for $\mathrm{Dom}(p)$. This composition must be in $P_{\prod_j X}$, since $\prod_{j\in J} p \in \prod_{j\in J} P$, and because Δ^* is a diagonal map, so that the composition is in fact one of generators of the product structure's plots.

Given any generator $c \circ \pi_j$ for the coplots of the product structure on $\prod_{j\in J} X$, where $c \in C$ and π_j is the jth canonical projection for some $j \in J$, $(c \circ \pi_j) \circ \Delta = c \circ (\pi_j \circ \Delta) = c \in C$. Therefore the diagonal map Δ is smooth by theorem 2.15, which assures us that we need only test the smoothness of maps on generating sets. □

Having access to these constructions, we will now move on to the demonstration of several topological properties of these spaces following the naming convention used for

Frölicher spaces in [2]. There are more useful ways to define these constructions, and more constructions which **Wat** is stable under, but as previously mentioned the discussion of these and other categorical questions will take place in a sequel.

3 The Topology of Watts Spaces

Given that a Watts structure is composed of two distinguished sets of mappings, there are two different topologies that one can discuss on any space equipped with such a structure.

Definition 3.1 *Given a Watts space (X, P, C), the* ***Plot Topology*** *is the final topology on X determined by P.*

When discussing Souriau spaces, which only come with a set of plots, this topology is the natural choice. In the study of Souriau spaces, it is often referred to as the D-topology.

Definition 3.2 *Given a Watts space (X, P, C), the* ***Coplot Topology*** *is the initial topology on X determined by C. A Watts space will always be assumed to be equipped with its coplot topology.*

Just as before, this topology is the natural choice when discussing Sikorski spaces, which only come equipped with a family of coplots. This state of affairs is rather aesthetically unpleasant; it would be preferable that these topologies happen to coincide, as such a coincidence removes any decision making on our part about which topology should be used.

Definition 3.3 *A Watts space (X, P, C) will be called* ***Balanced*** *if the plot and coplot topologies are equal to one another.*

This is not the case in general, for an example, consider the trivial structure on $\mathbb{R}$. Cursed with this freedom, it is traditional to choose the weakest of all available topologies for general considerations, and specify when a particular result relates to the stronger topology. The following result demonstrates that the coplot topology is the weaker of the two in general, justifying it being the topology always given to our Watts space unless otherwise specified.

Theorem 3.4 *The coplot topology is always coarser than the plot topology for any Watts space (X, P, C).*

Proof. Let $U \subset X$ be open in the coplot topology for some Watts space (X, P, C), $p \in P$, and suppose that $x \in p^{-1}(U)$. Then $p(x) \in U$, and since this set is a member of the initial topology determined by C, there must exist a finite subset $\{c_j\}_{j=1}^n \subset C$ and a accompanying finite collection of open sets $\{V_j \subset \mathbb{R}\}_{j=1}^n$ such that

$$p(x) \in \bigcap_{j=1}^{n} c_j^{-1}(V_j) \subset U. \tag{7}$$

Applying the preimage with respect to p to this containment yields us

$$\implies p^{-1}\big(p(x)\big) \subset p^{-1}\left(\bigcap_{j=1}^{n} c_j^{-1}(V_j)\right) = \bigcap_{j=1}^{n}(c_j \circ p)^{-1}(V_j) \subset p^{-1}(U). \tag{8}$$

Now since $(c_j \circ p)$ is a smooth function on a finite dimensional euclidean space for each $j \in \{1, 2, ..., n\}$ by the Watts axiom, they are also continuous. It is trivially true that $x \in p^{-1}\big(p(x)\big)$, and the intersection in the center of the containment chain above must be an open subset of $\mathrm{Dom}(p)$ by the continuity of the functions $\{c_j \circ p | j \in J\}$. This means that:

$$\implies x \in \bigcap_{j=1}^{n}(c_j \circ p)^{-1}(V_j) \subset p^{-1}(U). \tag{9}$$

Since $x \in p^{-1}(U)$ was chosen arbitrarily, the containment above implies that $p^{-1}(U) \subset \mathrm{Dom}(p)$ is an open set. This must be true for all $p \in P$, because the p in this argument was chosen arbitrarily as well, therefore U is a member of the plot topology. ☐

Since the plot topology is the finest on our space for which the plots are continuous, and the coplot topology is contained within it, the plots must be continuous in the coplot topology. Likewise The coplot topology is the coarsest topology for which the coplots are continuous, and the plot topology contains it, therefore the coplots are continuous when our space is given the plot topology. This means that regardless of the status of the topologies, whether they coincide, or a choice has been made, the coplots and plots will always be continuous maps.

Definition 3.5 *A topology on X is called **Smoothly Regular** with respect to a Watts space (X, P, C) when it satisfies the property that if $K \subset X$ is closed and $x \in X - K$, then there exists a $c \in C$ such that Ran$(c) \subset [0, 1]$, $c(x) = 1$, and $c|_K = 0$.*

A pleasant fact we shall soon see is that all Watts spaces are smoothly regular. This is due to the fact that we've defined our topology in terms of a set of real valued functions on our space which are closed with respect to compositions with smooth maps, together with the fact that $\mathbb{R}$ is smoothly normal and Hausdorff. This result was understood by Sikorski, who stated it for the category bearing his name [6].

Theorem 3.6 *For any disjoint closed sets $K, Q \subset \mathbb{R}$, there exists a smooth map $f : \mathbb{R} \to \mathbb{R}$ having the properties that Ran$(f) \subset [0, 1]$, $f|_K = 1$, and $f|_Q = 0$. In other words, $\mathbb{R}$ is smoothly normal.*

Proof. Let $K, Q \subset \mathbb{R}$ be two disjoint, closed sets. Then $\{\mathbb{R} - K, \mathbb{R} - Q\}$ is an open cover of $\mathbb{R}$, and theorem 2.18 of [5] ensures the existence of smooth partitions of unity subordinate to this cover. This means we can be certain of the existence of a pair of nonnegative smooth maps $\{f_Q, f_K\}$ such that $f_K|_Q = 0$, $f_Q|_K = 0$, and $f_K + f_Q = 1$. Nonegativity and $f_K + f_Q = 1$ together imply that $\mathrm{Ran}(f_K) \subset [0, 1]$. $f_K + f_Q = 1$ and $f_Q|_K = 0$ together imply that $f_K|_K = 1$. Therefore ,there indeed exists a smooth function f_K on $\mathbb{R}$ such that $f_K|_K = 1$, $f_K|_Q = 0$, and $\mathrm{Ran}(f_K) \subset [0, 1]$. ☐

Theorem 3.7 *All Watts spaces are smoothly regular.*

Proof. Let $K \subset X$ be closed in the coplot topology of a Watts space (X, P, C), meaning that $X - K$ is a member of the coplot topology. Therefore for any $x \in X - K$ there is a finite set of functions $\{c_j\}_{j=1}^n \subset C$ and accompanying open sets $\{U_j \subset \mathbb{R}\}_{j=1}^n$ such that:

$$x \in \bigcap_{j=1}^{n} c_j^{-1}(U_j) \subset X - K. \tag{10}$$

By the smooth normality of $\mathbb{R}$, and the fact that is it Hausdorff, we can be sure that for every closed set $\mathbb{R} - U_j \subset \mathbb{R}$, and any disjoint closed set such as the singleton determined by the point $c_j(x) \in U_j \subset \mathbb{R}$, there exists a smooth function $f_j : \mathbb{R} \to \mathbb{R}$ such that $\mathrm{Ran}(f_j) \subset [0, 1]$, $f_j(c_j(x)) = 1$, and $f_j|_{\mathbb{R}-U_j} = 0$.

From this it follows that $(f_j \circ c_j)$ is bounded above by $(f_j \circ c_j)(x) = f_j(c_j(x)) = 1$ for each $j \in \{1, 2, ..., n\}$. Moreover, because the intersection is a subset of $X - K$, for every $y \in K$ there must be some $j \in \{1, 2, ..., n\}$ such that $y \notin c^{-1}(U_j)$, which implies that $c_j(y) \in \mathbb{R} - U_j$. This means that for any $y \in K$, there is a $j \in \{1, 2, ..., n\}$ such that $(f_j \circ c_j)(y) = f_j(c_j(y)) = 0$. Finally, this all means that:

$$c = \prod_{j=1}^{n} (f_j \circ c_j), \tag{11}$$

is a function with the property that $c(x) = 1$, and $c(y) = 0$ for any $y \in K$. This means that it satisfies $c|_K = 0$. Moreover, since $\{c_j\}_{j=1}^n \subset C$, the functions $\{f_j\}_{j=1}^n$ are C^∞, multiplication of finitely many real numbers is C^∞, and c was formed by composing all of these maps, it must be that $c \in C$ by the first Sikorski axiom. Finally, that $\mathrm{Ran}(c) \subset [0, 1]$ can be observed from the fact that $(f_j \circ c_j)$ is nonnegative for each $j \in \{1, 2, ..., n\}$, and that $(f_j \circ c_j)(y) \leq (f_j \circ c_j)(x)$ for each $j \in \{1, 2, ..., n\}$ and each $y \in X$, which together imply that $\prod_{j=1}^{n}(f_j \circ c_j)(y) \leq \prod_{j=1}^{n}(f_j \circ c_j)(x)$ for each $y \in X$. Thus c is indeed the member of C with our desired properties, and so (X, P, C) is smoothly regular. $\square$

Having this strong property on hand permits the simplification of several proofs, such as the proof that smoothness implies continuity.

Theorem 3.8 *Smooth maps are continuous.*

Proof. Let $f : A \to Y$ be a smooth map for Watts spaces (X, P_X, C_X) and (Y, P_Y, C_Y) with $A \subset X$. By definition, there must be some smooth $f^* : X \to Y$ such that $f^*|_A = f$. Suppose that $U \subset Y$ was an open set. For every $x \in {f^*}^{-1}(U)$, it must be that $x \in {f^*}^{-1}(y)$ for some $y \in U$. Every Watts space is smoothly regular by the previous result, meaning there is a $c \in C_Y$ so that $c(y) = 1$, $c|_{Y-U} = 0$, and $\mathrm{Ran}(c) \subset [0, 1]$, and so $c \circ f^* \in C_X$ by f^*'s smoothness such that $(c \circ f^*)|_{X-f^{-1}(U)} = c|_{Y-U} \circ f|_{f^{-1}(U)} = 0$ and $(c \circ f)(x) = c(y) = 1$.

This means that for every $x \in {f^*}^{-1}(U)$, there is a $c \circ f^* \in C_X$ such that $x \in (c \circ f^*)^{-1}(0, \infty) \subset {f^*}^{-1}(U)$, therefore ${f^*}^{-1}(U) \subset X$ is an open set and so f^* must be continuous. Since f is a restriction of a continuous map f^*, it must be continuous when A is given the subspace topology from X. $\square$

The result above does not require that we invoke smooth regularity, it could have been proven immediately after introducing the coplot topology, but access to this property leads to a shorter, simpler proof. Before moving on to the rest of the separation properties, we will need to convince ourselves that the properties mandated in the axioms of Souriau and Sikorski for plots and coplots extend to all smooth maps.

Theorem 3.9 *Smooth maps inherit the properties required of the plots and coplots:*

1) All constant maps are smooth.

2) If $\{f_j : X \to Y_j\}_{j\in J}$ is a set of smooth maps $g : \prod_{j\in J} Y_j \to Z$ is a smooth map, then $g(f_j)_{j\in J} : X \to Z$ is a smooth map.

3) If $f : X \to Y$ is a function such that there is an open cover $\{U_j\}_{j\in J}$ of X and a set of smooth maps $\{f_j : X \to Y\}_{j\in J}$, such that $f|_{U_j} = f_j|_{U_j}$ for every $j \in J$, then f is smooth.

Proof. 1) Let $f : X \to Y$ be constant, meaning that $f(x_1) = f(x_2)$ for all $x_1, x_2 \in X$. Then for any $c \in C_Y$ or any $p \in P_X$, $(c \circ f)(x_1) = c(f(x_1)) = c(f(x_2)) = (c \circ f)(x_2)$ for all $x_1, x_2 \in X$, and $(f \circ p)(t_1) = f(p(t_1)) = f(p(t_2)) = (f \circ p)(t_2)$ for all $t_1, t_2 \in \mathrm{Dom}(p)$. Therefore both $c \circ f$ and $f \circ p$ are constant, and so $c \circ f \in C_X$ and $f \circ p \in P_Y$ by the first Sikorski and Souriau axioms, and therefore f is smooth.

2) Let $\{f_j : X_j \to Y_j\}_{j\in J}$ be a set of smooth maps and $g : \prod_{j\in J} Y_j \to Z$ be a smooth map. Then $g(f_j)_{j\in J} = g \circ \left(\prod_{j\in J} f_j\right) \circ \Delta$, therefore the smoothness of cartesian products of smooth maps, the smoothness of the diagonal map $\Delta : X \to \prod_{j\in J} X$, and the smoothness of compositions of smooth functions ensures that $g(f_j)_{j\in J}$ is also smooth.

3) Let $f : X \to Y$ be a function such that there is an open cover $\{U_j\}_{j\in J}$ of X and a set of smooth maps $\{f_j : X \to Y\}_{j\in J}$, such that $f|_{U_j} = f_j|_{U_j}$ for every $j \in J$. This means that for all $c \in C_Y$, and all $p \in P_X$, we have the containments $c \circ f_j \in C$ and $f_j \circ p \in P$ for each $j \in J$. This implies the existence of an open cover $\{U_j\}_{j\in J}$ of X such that $(c \circ f)|_{U_j} = c \circ (f|_{U_j}) = c \circ (f_j|_{U_j}) = (c \circ f_j)|_{U_j}$ for all $j \in J$, thus $c \circ f \in C_X$ by the third Sikorski axiom. On the other hand, p's continuity ensure that $\{p^{-1}(U_j)\}_{j\in J}$ is an open cover of $\mathrm{Dom}(p)$ such that $(f \circ p)|_{p^{-1}(U_j)} = f|_{U_j} \circ p|_{p^{-1}(U_j)} = f_j|_{U_j} \circ p|_{p^{-1}(U_j)} = (f_j \circ p)|_{p^{-1}(U_j)}$ for each $j \in J$, thus $f \circ p \in P_Y$, and so f is smooth by definition. □

Smooth regularity also gives us a first method of testing, or perhaps ensuring, the balance of a Watts space: Determine if or arrange that the plot topology is smoothly regular.

Theorem 3.10 *A Watts space is balanced if and only if its plot topology is smoothly regular.*

Proof. First, assume that (X, P, C)'s plot topology is smoothly regular. By theorem 3.4, we know that the initial topology determined by C is contained in the final topology determined by P for any Watts space. Moving on to the opposing containment, let $U \subset X$ be open in the final topology determined by P. Then $X - U$ is a closed set, and any $x \in U$ is not contained in $X - U$, which by our hypothesis implies the existence of a $c \in C$ with $\mathrm{Ran}(c) \subset [0, 1]$ such that $c(x) = 1$ and $c|_{X-U} = 0$. This means that for the open set

$(0, \infty)$, $x \in c^{-1}\big((0, \infty)\big) \subset U$, and since $x \in U$ was chosen arbitrarily, this means that U is open in the initial topology determined by C.

Now instead assume that the plot and coplot topologies of our Watts space (X, P, C) coincide, so that its plot topology is equal to its coplot topology. Then the plot topology must be smoothly regular, owing to the facts that the coplot topology must be smoothly regular by theorem 3.7, and that the two topologies are equal. $\square$

Returning to the question of separation axioms, recall that on any smooth manifold, the Kolmogorov, Fréchet, and Hausdorff separation axioms are equivalent. This can be proven quickly from metric space structure of any euclidean space and the fact that all smooth manifolds are locally euclidean, but it can also be shown to hold for Watts spaces despite the fact that they are not locally-euclidean in general.

Definition 3.11 *A Watts space (X, P, C) is **Smoothly Hausorff** if C separates X, that is, for any distinct $x_1, x_2 \in X$ there exists a $c \in C$ such that $c(x_1) \neq c(x_2)$.*

A space which is smoothly Hausdorff is also Hausdorff, as we will come to see that the point separation in the definition of smoothly Hausdorff, the Hausdorffness of $\mathbb{R}$, and the continuity of coplots come together to imply the existence of disjoint open sets for any distinct points. Less obvious is how the weakest separation axiom could imply a strengthened version of the second, but just as in the case of smooth manifolds, we are still able to extract nice properties for these spaces from their close contact with $\mathbb{R}$.

Theorem 3.12 *The properties Hausdorff, Smoothly Hausdorff, Fréchet, and Kolmogorov are all equivalent on Watts spaces.*

Proof. We begin by noting that any space which is Hausdorff must be Fréchet, so all Hausdorff Watts spaces are Fréchet. We follow this up with a similar realization that any space which is Fréchet must be Kolmogorov.

Now, assume that (X, P, C) is a Kolmogorov Watts space. Then for any distinct points $x_1, x_2 \in X$, there is an open set $U \subset X$ such that $x_1 \in U$ and $x_2 \notin U$. Therefore $K = X - U$ is a closed set containing x_2, and not containing x_1, and so by theorem 3.7 there exists a $c \in C$ such that $c|_K = 0$ and $c(x_1) = 1$. Therefore $c(x_1) = 1 \neq 0 = c(x_2)$, and so our Watts space is smoothly Hausdorff.

If we're instead given a smoothly Hausdorff Watts space (X, P, C), then the Hausdorffness of $\mathbb{R}$, along with the fact that C separates points in X, and that members of C are continuous, together imply the existence of disjoint open neighborhoods for any distinct points. Therefore smoothly Hausdorff Watts spaces are indeed Hausdorff, and so the four separation properties are equivalent on Watts spaces. $\square$

In order to demonstrate the next separation result, we will first need to demonstrate that the set of smooth euclidean valued maps is closed with respect to sums of locally finite subcollections.

Definition 3.13 *Let X be a topological space, and F a collection of maps all taking on values in the same euclidean space and defined on X, such collections will be called*

Locally Finite *if for any $x \in X$ there is an open set $U \subset X$ containing x such that at most a finite number F's members have non-zero restrictions for this set U.*

Theorem 3.14 *Let (X, P, C) be a Watts space. Given a locally finite collection of smooth functions C^*, all having the same euclidean space as their codomain and X as their domain, there is a smooth map c whose value at every point is equal to the sum of all members of C^* which are nonzero there, or zero should all of C^* vanish there.*

Proof. Let C^* be a locally finite collection of smooth functions on the Watts space (X, P, C) all sharing the same euclidean space for a codomain, meaning that for every $x \in X$ there is an open set $U_x \subset X$ containing x such that all members of C^* besides a finite (potentially empty) subset $\{c_{x,j}\}_{j=1}^{n_x}$ restrict to zero on U_x. Clearly $\{U_x\}_{x \in X}$ is an open cover of X, therefore we may attempt to define a function's value for every $y \in X$ in the following way:

$$c(y) = \sum_{j=1}^{n_x} c_{x,j}(y) \text{ where } y \in U_x \quad \forall y \in X. \tag{12}$$

To see that this function is indeed well-defined, let $y \in U_{x_1} \cap U_{x_2}$ for some $x_1, x_2 \in X$, and let $\{c_{1,j}\}_{j=1}^{n_1}$ and $\{c_{2,j}\}_{j=1}^{n_2}$ be the finite subsets of C^* which have nonzero restrictions to U_{x_1} and U_{x_2}, respectively. If $c_{1,j}(y) \neq 0$ then $c_{1,j}$ must be a member of $\{c_{2,j}\}_{j=1}^{n_2}$ because it will be nonzero when restricted to U_{x_2}, because it contains y. The exact same argument works to show that any $c_{2,j}$ for which $c_{2,j}(y) \neq 0$ must be in $\{c_{1,j}\}_{j=1}^{n_1}$. This means that the collections $\{c_{1,j}\}_{j=1}^{n_1}$ and $\{c_{2,j}\}_{j=1}^{n_2}$ differ only by functions which vanish at y, and so the sum of all outputs formed from either collection at y must be the same. Therefore the value of this function is well-defined at every point in the domain.

Now that we are certain that this definition for a sum is uniquely valued at all points, let us also note that $\{U_x\}_{x \in X}$ is an open cover of X for which $c|_{U_x}$ is equal to a sum of a finite set of smooth functions by its definition, and therefore it is a smooth function by theorem 3.9. Therefore summation over locally finite collections of functions is not only well-defined, the class of all smooth maps from a Watts space to a Euclidean space is closed with respect to the summation of its locally finite subcollections. □

Definition 3.15 *Given a Watts space (X, P, C), and an open cover $\{U_j\}_{j \in J}$, a* ***Smooth partition of unity subordinate to*** *$\{U_j\}_{j \in J}$ is a locally-finite collection of coplots $\{c_\alpha\}_{\alpha \in A} \subset C$, such that for all $\alpha \in A$ there is a $j \in J$ so that $c_\alpha|_{X - U_j} = 0$, $c_\alpha(X) \subset [0, 1]$ for every $\alpha \in A$, and:*

$$\sum_{\alpha \in A} c_\alpha = 1. \tag{13}$$

This definition for smooth partition of unity is not exactly the one commonly stated for smooth manifolds, found in [5]. It does not refer to the supports of its members, and the partition is not necessarily indexed over the same set as the cover to which it is subordinate. Despite these modifications, this definition is still perfectly adequate not only for the separation properties still to be demonstrated, it is also sufficient for a generalization of Whitney's approximation theorem to these new spaces.

Definition 3.16 *Given a Watts space (X, P, C), a topology on X is* ***Smoothly Paracompact*** *if there exists smooth partitions of unity subordinate to any open cover in that topology.*

Theorem 3.17 *If a Watts space is smoothly paracompact, then it is paracompact.*

Proof. Let (X, P, C) be a smoothly paracompact Watts space, and let $\{V_j\}_{j\in J}$ be an open cover of X. Then there exists a smooth partition of unity $\{c_\lambda\}_{\lambda\in L} \subset C$ subordinate to $\{V_j\}_{j=1}$. That $\{c_\lambda^{-1}\big((0,\infty)\big)\}_{\lambda\in L}$ is an open cover of X follows immediately from requirement that the partition's sums to one:

$$\sum_{\lambda\in L} c_\lambda = 1.$$

Moreover, since for every $\lambda \in L$ there is a $j \in J$ such that $c_\lambda|_{X-V_j} = 0$, it must be that $c_\lambda^{-1}\big((0,\infty)\big) \subset V_j$. Therefore $\{c_\lambda^{-1}\big((0,\infty)\big)\}_{\lambda\in L}$ is a refinement of $\{V_j\}_{j\in J}$. The local finiteness of $\{c_\lambda\}_{\lambda\in L}$ guarantees that $\{c_\lambda^{-1}\big((0,\infty)\big)\}_{\lambda\in L}$ must be locally finite, therefore (X, P, C) is paracompact, since any open cover $\{V_j\}_{j\in J}$ possesses a locally finite refinement $\{c_\lambda^{-1}\big((0,\infty)\big)\}_{\lambda\in L}$. $\square$

The name smooth paracompactness is somewhat justified by its ability to imply its continuous namesake. Moving along, we now turn our attention to the smoothly normal separation property.

Definition 3.18 *Given a Watts space (X, P, C), a topology on X is* ***Smoothly Normal*** *if for any disjoint closed sets $K_1, K_2 \subset X$, there is a $c \in C$ such that $Ran(c) \subset [0, 1]$, $c|_{K_1} = 1$, and $c|_{K_2} = 0$.*

Theorem 3.19 *Smoothly paracompact Watts spaces are smoothly normal.*

Proof. Let $K, Q \subset X$ be two closed subsets of a Watts space (X, P, C), such that $K \cap Q = \emptyset$. $\{X - Q, X - K\}$ is an open cover of X, therefore by smooth paracompactness there is a smooth partition of unity $\{c_Q, c_K\} \subset C$ subordinate to this open cover. Without loss of generality, it can be taken to have the two members specified, because local finiteness and theorem 3.14 guarantee that we can add all members of a partition subordinate to the same open set to arrive at a partition with only two members. Since $c_Q|_Q, c_K|_K = 0$ by definition 3.15, and $(c_Q + c_K)|_Q = 1$ by the same definition, it must be that $c_Q|_K = 1$, being that definition 3.15 goes on to confine c_Q's range to $[0, 1]$. Therefore C_Q is exactly the function we seek to separate K and Q, and so X is indeed smoothly normal. $\square$

Not only does smooth paracompactness imply smooth normality on these spaces, but access to smooth partitions of unity allows for the approximation of continuous mappings by smooth ones.

Theorem 3.20 *Let (X, P, C) be a smoothly paracompact Watts space, $f : X \to \mathbb{R}^m$ for $m \in \mathbb{N}$ and $\delta : X \to (0, \infty)$ be two continuous functions, and allow $A \subset X$ to be a closed set on which f is smooth. There exists is a smooth function $c : X \to \mathbb{R}^m$ such that $|f(x) - c(x)| < \delta(x)$ for all $x \in X$, and such that $c|_A = f|_A$.*

Proof. Let $f : X \to \mathbb{R}^m$ and $\delta : X \to (0, \infty)$ be two continuous functions on a smoothly paracompact Watts space (X, P, C), let $A \subset X$ be closed, and let q be a smooth map such that $f|_A = q|_A$. Then because f, q, and δ are continuous by theorem 3.8 and by hypothesis, the set U_A of all $x \in X$ such that $|f(x) - q(x)| < \delta(x)$ is open, and contains A.

For every $y \in X - A$, there is an open set U_y defined to be the intersection of $X - A$ with the set of all $x \in X$ such that $|f(y) - f(x)| < \delta(x)$. $\{U_A, U_y | y \in X - A\}$ is an open cover of X, therefore the smooth paracompactness of (X, P, C) implies that that there is a smooth partition of unity $\{c_A, c_j | j \in J\} \subset C$ subordinate to this open cover (c_A with $c_A|_{X-U_A} = 0$ must be a member because U_A is the only member of the open cover which contains the points in A by construction). Therefore $\{c_A q, c_j f(y_j) | j \in J\}$ is a locally-finite collection of smooth functions, where y_j is a point in $X - U_A$ indexing a cover element U_{y_j} for which $c_j|_{X-U_{y_j}} = 0$ for each $j \in J$. This allows us to employ theorem 3.14 to conclude that the locally finite sum:

$$c = qc_A + \sum_{j \in J} f(y_j)c_j \tag{14}$$

is a well defined smooth function. Furthermore, by its construction, for any $x \in X$

$$|f(x) - c(x)| = \left| f(x) - q(x)c_A(x) - \sum_{j \in J} f(y_j)c_j(x) \right|$$

$$= \left| f(x) \left(c_A(x) + \sum_{j \in J} c_j(x) \right) - q(x)c_A(x) - \sum_{j \in J} f(y_j)c_j(x) \right|$$

$$= \left| (f(x) - q(x))c_A(x) + \sum_{j \in J} (f(x) - f(y_j))c_j(x) \right|$$

$$\leq |f(x) - q(x)|c_A(x) + \sum_{j \in J} |f(x) - f(y_j)|c_j(x)$$

$$< \delta(x)c_A(x) + \sum_{j \in J} \delta(x)c_j(x) = \delta(x).$$

We are justified in applying the inequalities with δ because the elements of the partition of unity ensure that the only nonzero values of the summands occur inside the open sets where they obey the inequality. Thus c not only is a smooth map with $c|_A = q|_A = f|_A$, which we can see is because $c_j|_A = 0$ for each $j \in J$ on account of A being outside of every member of $\{U_{y_j}\}_{j \in J}$ by construction, it also obeys the approximation inequality $|f(x) - c(x)| < \delta(x)$ for all $x \in X$. $\square$

The proof of this special case of Whitney's Approximation Theorem presented here is essentially identical to the one found in Lee's *Introduction to Smooth Manifolds* [5], but for the alternative space of interest, and the minor variation on the definition for smooth partition of unity.

Theorem 3.21 *Let $A \subset X$ be a closed subset of a smoothly paracompact Watts space (X, P, C), and let $f : A \to \mathbb{R}$ be a smooth map, then there is a coplot $c \in C$ which is bounded if f is bounded, and which restricts to f on A.*

Proof. To begin with, if $f : A \to \mathbb{R}$ is smooth, then by theorem 3.8 it must also be continuous. By theorem 3.19, we know that any smoothly paracompact Watts space (X, P, C) must be smoothly normal. The continuity of members of C, and the Hausdorffness of $\mathbb{R}$ then quickly imply that X must be normal.

The Tietze Extension Theorem then guarantees that there is a continuous map k such that $k|_A = f$ and k is bounded if f is. Then by the Whitney approximation theorem and theorem 2.9, there must be a $c \in C$ such that $c|_A = k|_A = f$, and such that $|c(x)-k(x)| < 1$ for all $x \in X$ meaning $c \in C$ can be chosen to be bounded if f is. $\square$

These last two theorems on smooth extensions and smooth approximations of continuous functions confirm that our visual experience of continuity, that of being very close to smoothness, is not a misleading one. It is something we can rely on for smoothly paracompact spaces in general, even when we are beyond hope of visualizing them.

Theorem 3.22 *There is a smooth function $h : \mathbb{R} \to [0, 1]$ for which $h|_{(-\infty,1/4]} = 0$ and $h|_{[3/4,\infty)} = 1$.*

Proof. Consider the functions:

$$b(x) = \begin{cases} e^{\frac{1}{(x-1/4)(x-3/4)}} & x \in (1/4, 3/4) \\ 0 & x \in \mathbb{R} - (1/4, 3/4) \end{cases} \tag{15}$$

$$h(x) = \frac{1}{\int_{-\infty}^{3/4} b(y)dy} \int_{-\infty}^{x} b(u)du \tag{16}$$

The smoothness of b follows from the chain and product rules, which tell us that the derivatives of the nonzero branch will always be some rational function of x times the original exponential term, which itself decays to zero as x approaches $1/4$ or $3/4$ faster than any rational function can grow at either of those two points.

h must be smooth since it is defined as the integral of a smooth function with compact support. Note that for all $x \in (-\infty, 1/4]$, we have that $b(x) = 0$, and so consequently $h(x) = 0$. Similarly, for all $x \in [3/4, \infty)$, we also have that $b(x) = 0$, meaning that

$$h(x) = \frac{1}{\int_{-\infty}^{3/4} b(y)dy} \int_{\infty}^{x} b(u)du = 1 + \frac{1}{\int_{-\infty}^{3/4} b(y)dy} \int_{3/4}^{x} b(u)du = 1.$$

Therefore h is indeed a smooth map satisfying our desirata. $\square$

Theorem 3.23 *The connected components of a balanced Watts space are smoothly path-connected.*

Proof. It is sufficient to demonstrate that a balanced Watts space (X, P, C) is locally smoothly path-connected. This is because the set of all smooth path components will then be is a disjoint open cover of X, and the existence of non-empty intersections would allow for the use of the smooth concatenation used later in this proof to show that the components being intersected weren't maximal, a contradiction. This allows us to conclude that each connected component of our space is equal to one of these smooth path components.

To proceed with the proof of local smooth path-connection, let $x \in X$, and let $Q_x \subset X$ be the set of points $y \in X$ such that there is a member $p^* \in P$ for which $\mathrm{Dom}(p^*) = \mathbb{R}$ and $y, x \in p^*(\mathbb{R})$. Let $p \in P$ be such that $\mathrm{Ran}(p) \cap Q_x \neq \emptyset$, and let $z \in p^{-1}(Q_x) \subset \mathrm{Dom}(p)$. Now $\mathrm{Dom}(p)$ is an open subset of a euclidean space, and so there is an open metric ball $B_\varepsilon(z)$ of some radius $\varepsilon > 0$ centered at z and contained in $\mathrm{Dom}(p)$. Let $v \in B_\varepsilon(z)$, and note that the straight line $\gamma_1(t) = v + t(z - v)$ is a smooth curve in $\mathrm{Dom}(p)$ with $\gamma_1(0) = v$ and $\gamma_1(1) = z$, so that $p_1 = p \circ \gamma_1$ is a plot by the second Souriau axiom with $p_1(0) = p(v)$ and $p_1(1) = p(z)$. $p(z) \in Q_x$, and so without loss of generality there is a plot $p_2 \in P$ with $p_2(0) = p(z)$, and $p_2(1) = x$ by our construction of Q_x. Consider the map:

$$p^*(t) = \begin{cases} p_1 \circ h(2t) & t \in \left(-\infty, \frac{1}{2}\right] \\ p_2 \circ h(2t-1) & t \in \left[\frac{1}{2}, \infty\right) \end{cases} \tag{17}$$

For which $p^*(0) = p_1(0) = p(v)$ and $p^*(1) = p_2(1) = x$, where h is the function defined in theorem 3.22. Due to the fact that $h|_{(-\infty,1/4]} = 0$ and $h|_{[3/4,\infty)} = 1$, and that $p_1(1) = p(z) = p_2(0)$, p^* is constant and equal to $p(z)$ on the open interval $\left(\frac{3}{8}, \frac{5}{8}\right)$. It is equal to the plots $p_1 \circ h$ and $p_2 \circ h$ on the intervals $\left(-\infty, \frac{1}{2}\right)$ and $\left(\frac{1}{2}, \infty\right)$ respectively. Therefore by the first and second Souriau axioms and the fact that $\left\{\left(-\infty, \frac{1}{2}\right), \left(\frac{3}{8}, \frac{5}{8}\right), \left(\frac{1}{2}, \infty\right)\right\}$ is an open cover of $[0, 1]$ on which p^*'s restrictions are equal to constant maps or restrictions of members of P, we may conclude that $p^* \in P$.

Therefore $p(B_\varepsilon(z)) \subset Q_x$ by Q_x's definition, and so $B_\varepsilon(z) \subset p^{-1}(Q_x)$. This means that $p^{-1}(Q_x)$ is an open set, which since the only requirement on p was that $p^{-1}(Q_x) \neq \emptyset$, we can conclude that Q_x is open in the plot topology. Since our Watts space is balanced by hypothesis, Q_x is also open in the coplot topology with which our space has been equipped. Therefore our Watts space (X, P, C) is indeed locally, smoothly path-connected. $\square$

This proof is adapted from that of lemma 1.8 in [4], where Laubinger demonstrated that Souriau spaces, which always bear their plot topology, are locally-path connected. The modifications introduced here are the halting functions, used to ensure that the concatenation of the two paths was smooth, and the requirement that the plot and coplot topologies coincide, which allows for the openness test used in the argument to be valid.

4 Conclusion

The category of Watts spaces and smooth maps, bearing the object class suggested by Watts, and which contains the Souriau, Sikorski, and Frölicher spaces as full subcategories, was described. It was shown that the Kolmogorov and Hausdorff separation axioms are equivalent on Watts spaces. The existence of smooth approximations to continuous maps into euclidean spaces, and extensions to smooth maps into euclidean spaces was also shown, along with a condition which is sufficient to guarantee that connected components of the objects belonging to this category are smoothly path-connected.

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Bryce Virgin
California State Polytechnic University, Pomona
3801 W Temple Ave.
Pomona, CA 91768
E-mail: `bgvirgin@cpp.edu`

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