# Local Entropy and $L^q$-Dimensions of Measures in Doubling Metric Spaces

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**Abstract -** We define restricted entropy and $L^q$-dimensions of measures in doubling metric spaces and show that these definitions are consistent with the monotonicity of $L^q$-dimensions. This provides a correct proof for a theorem considering the relationships between local entropy and $L^q$-dimensions in a paper by Käenmäki, Rajala and Suomala, the original proof of which makes use of a slightly erroneous proposition.

**Keywords :** multifractal analysis; local $L^q$-dimension; local entropy dimension

**Mathematics Subject Classification** (2020) **:** 28A80; 54E50

## 1 Introduction

In multifractal analysis, one is interested in the behaviour of the local dimension map

$$x \mapsto \dim_{\text{loc}}(\mu, x) = \lim_{r \downarrow 0} \frac{\log \mu(B(x,r))}{\log r},$$

where $\mu$ is often a fractal type measure, and the level sets of the local dimension map exhibit fractal scaling according to a spectrum of dimensions. These types of measures are often called *multifractal measures* although a precise definition is avoided. In particular one is interested in properties of the level sets

$$E_\alpha = \{x : \dim_{\text{loc}}(\mu, x) = \alpha\},$$

mainly the Hausdorff and packing dimensions of said sets. Multifractal formalism is a heuristic principle with origins in physics literature [3] which states that the Hausdorff and packing dimensions of these levels sets are given by the Legendre transform of the $L^q$-spectrum $\tau_q(\mu)$ of the measure (see Section 2.2 for the precise definitions), that is

$$\dim_H E_\alpha = \dim_P E_\alpha = \inf_{q \in \mathbb{R}} \{q\alpha - \tau_q(\mu)\}.$$

Recently, multifractal analysis has received quite a bit of interest due to its many applications in different fields adjacent to mathematics. Multifractals have proved to be a useful tool in understanding turbulence in fluids [3], complex networks [10], economics [5] and medical imaging [6] just to name a few examples. Mathematically the problem is also interesting since in applications the definitions are quite non-standard and often establishing the multifractal formalism in a mathematically rigorous way requires careful investigation of the measures in question.

The most simple and classical example of a class of measures for which the multifractal formalism is known to hold are self-similar measures under the strong separation condition [1, 2]. The results presented by Käenmäki, Rajala and Suomala in [8, 9] provide a generalisation of this classical situation into doubling metric spaces. In addition, the authors provide a local variant of multifractal analysis, which is also useful in the Euclidean case.

This paper aims to correct a small inaccuracy in [8]. The authors claim in [8, Proposition 3.2] that the global $L^q$-spectrum can be calculated using partitions of the doubling metric space $X$ instead of packings (see Section 2.1 for definitions), which are used in the definition. However, the counterexample in Section 2 shows that this is indeed not the case. Our aim is to provide an alternate version of the statement using the restricted $L^q$-spectrum, which gives correct proofs for the results in [8] that make use of the incorrect proposition.

This paper is organised as follows: Section 2 gives a brief recap of the concepts we are working with and introduces some notation, as well as gives a counterexample to [8, Proposition 3.2]. In Section 3 we provide an alternate version of the erroneous proposition and give a proof for the statement. We conclude the paper by stating the main results in Section 4 and by giving a remark concerning the application of the theory in Section 5. For an introduction to multifractal analysis we refer to [1, 2].

## 2 Preliminaries

### 2.1 Notation

For constants, we use the notation $c = c(\ldots)$, meaning that the constant depends on the parameters listed inside the parentheses. Closed balls, with center $x \in X$ and radius $r$ are denoted by $B(x, r) = \{y \in X : d(x, y) \leq r\}$. For $M > 0$ and a ball $B = B(x, r)$ we use the abbreviation $MB = B(x, Mr)$, when the radius and center of the ball $B$ are fixed.

In this paper we always work in *doubling metric spaces* $(X, d)$, which means that there exists a constant $N = N(X)$ called the *doubling constant* of $X$, such that any closed ball in $X$, with centre $x$ and radius $r > 0$ can be covered with $N$ balls of radius $r/2$. The distance function $d$ is fixed for the space so we refer to $(X, d)$ simply as $X$.

Any countable collection $\mathcal{B}$ of pairwise disjoint closed balls is called a *packing* and if the centres of the balls are in a subset $A \subset X$ it is called a packing of $A$. For $\delta > 0$ the packing $\mathcal{B}$ is called a $\delta$-packing if each of the balls in $\mathcal{B}$ has a radius of $\delta$. A collection $\mathcal{Q}$ of non-empty subsets of $X$, such that every element of $X$ is contained in exactly one $Q \in \mathcal{Q}$ is called a *partition (of $X$)*.

Let $1 \leq \Lambda < \infty$. For $\delta > 0$, a countable partition $\mathcal{Q}$ of $X$ is called a *$(\delta, \Lambda)$-partition* if all of the sets $Q \in \mathcal{Q}$ are Borel sets and for each $Q \in \mathcal{Q}$ there exists a ball $B_Q$ such that $Q \subset \Lambda B_Q$ and the collection $\{B_Q : Q \in \mathcal{Q}\}$ is a $\delta$-packing. Usually we assume that $\Lambda$ has been fixed and only talk about $\delta$-partitions, since the choice of $\Lambda$ is often irrelevant.

Let $(\delta_n)_{n\in\mathbb{N}}$ be a decreasing sequence of real numbers so that there is $0 < c < 1$ for which

$$\delta_n < c^n \tag{2.1}$$

for all $n \in \mathbb{N}$ and

$$\lim_{n\to\infty} \frac{\log \delta_n}{\log \delta_{n+1}} = 1. \tag{2.2}$$

Here and hereafter we assume that for each $n \in \mathbb{N}$ we have fixed a $\delta_n$-partition that satisfies the previous assumptions and denote it by $\mathcal{Q}_n$. For $x \in X$, we denote the unique element of $\mathcal{Q}_n$ containing $x$ by $\mathcal{Q}_n(x)$. For $A \subset X$ we set $\mathcal{Q}_n(A) = \{Q \in \mathcal{Q}_n : A \cap Q \neq \emptyset\}$. For a fixed $\delta_n$-partition we silently assume that $\Lambda$ is the same for all $\delta_n$.

In the context of this paper, a *measure* always refers to a locally finite Borel regular (outer) measure defined on all subsets of $X$. The *support* of a measure $\mu$ is the smallest closed subset of $X$ with full $\mu$-measure and is denoted by $\mathrm{spt}(\mu)$.

Next we present a lemma which shows that it is possible to state the doubling property of the metric space $X$ in multiple equivalent ways. The proof of the lemma is a simple exercise [4].

**Lemma 2.1.** *For a metric space $X$, the following statements are equivalent*

1. *$X$ is doubling*
2. *There are $s > 0$ and $c > 0$ such that for all $R > r > 0$ any ball of radius $R$ can be covered by $c(r/R)^{-s}$ balls of radius $r$.*
3. *There are $s > 0$ and $c > 0$ such that if $R > r > 0$ and $\mathcal{B}$ is an $r$-packing of a closed ball of radius $R$, then the cardinality of $\mathcal{B}$ is at most $c(r/R)^{-s}$*
4. *For every $0 < \lambda < 1$ there is a constant $M = M(X, \lambda) \in \mathbb{N}$ satisfying the following: if $\mathcal{B}$ is a collection of closed balls of radius $\delta > 0$ so that $\lambda\mathcal{B}$ is pairwise disjoint, then there are $\delta$-packings $\{\mathcal{B}_1, \ldots, \mathcal{B}_M\}$ so that $\mathcal{B} = \bigcup_{i=1}^{M} \mathcal{B}_i$.*
5. *There is $M = M(X) \in \mathbb{N}$ such that if $A \subset X$ and $\delta > 0$, then there are $\delta$-packings of $A$, $\mathcal{B}_1, \ldots, \mathcal{B}_M$ whose union covers $A$.*

One final property referenced in this paper is the *density point property* for a measure $\mu$, which is said to hold if

$$\lim_{r\downarrow 0}\frac{\mu(A\cap B(x,r))}{\mu(B(x,r))}=1,$$

for $\mu$-almost all $x\in A$, whenever $A\subset X$ is $\mu$-measurable. We note that the property holds for every finite Borel measure in Euclidean spaces, but not necessarily in arbitrary doubling metric spaces.

### 2.2 Dimensions of measures

The *upper and lower local dimensions* of a measure $\mu$ at the point $x$ are given by

$$\overline{\dim}_{\mathrm{loc}}(\mu,x)=\limsup_{r\downarrow 0}\frac{\log\mu(B(x,r))}{\log r},$$
$$\underline{\dim}_{\mathrm{loc}}(\mu,x)=\liminf_{r\downarrow 0}\frac{\log\mu(B(x,r))}{\log r},$$

respectively. If the upper and lower dimensions agree, their common value is referred to as the *local dimension* of $\mu$ at $x$ and we write $\dim_{\mathrm{loc}}(\mu,x)=\overline{\dim}_{\mathrm{loc}}(\mu,x)=\underline{\dim}_{\mathrm{loc}}(\mu,x)$

For a bounded subset $A\subset X$, the *$L^q$-moment sum of $\mu$ on $A$ at scale $\delta$* is defined by

$$S_q(\mu,A,\delta)=\sup\{\sum_{B\in\mathcal{B}}\mu(B)^q\ :\ \mathcal{B}\text{ is a }\delta\text{-packing of }A\cap\operatorname{spt}(\mu)\}. \tag{2.3}$$

We then define the *(global) $L^q$-spectrum of $\mu$ on $A$* as

$$\tau_q(\mu,A)=\liminf_{\delta\downarrow 0}\frac{\log S_q(\mu,A,\delta)}{\log\delta}.$$

The definition given here is the same that is used in [8]. Notice that for $q\geq 0$ the definition remains unchanged if $A\cap\operatorname{spt}(\mu)$ is replaced with $A$ in (2.3). For $q\neq 1$ we define the *$L^q$-dimension of $\mu$ on $A$* by

$$\dim_q(\mu,A)=\tau_q(\mu,A)/(q-1).$$

If $X$ is bounded, we use the shorthand notation $\tau_q(\mu)=\tau_q(\mu,X)$ and $\dim_q(\mu)=\dim_q(\mu,X)$. For any $A\subset X$, with $\mu(A)>0$ we define the *(global) upper and lower entropy dimensions of $\mu$ on $A$* as

$$\overline{\dim}_1(\mu,A)=\limsup_{\delta\downarrow 0}⨍_A\frac{\log\mu(B(y,\delta))}{\log\delta}\mathrm{d}\mu(y),$$
$$\underline{\dim}_1(\mu,A)=\liminf_{\delta\downarrow 0}⨍_A\frac{\log\mu(B(y,\delta))}{\log\delta}\mathrm{d}\mu(y)$$

respectively. Here and hereafter we use the notation

$$\fint_A f(x)\mathrm{d}\mu(x) = \mu(A)^{-1}\int_A f(x)\mathrm{d}\mu(x).$$

If the values of the dimensions agree then the common value is referred to as the *(global) entropy dimension of* $\mu$ *on* $A$ and is denoted by $\dim_1(\mu, A)$.

From the above definitions we derive the local variants. The *local* $L^q$*-spectrum of* $\mu$ *at* $x \in spt(\mu)$ is defined as

$$\tau_q(\mu, x) = \lim_{r\downarrow 0}\tau_q(\mu, B(x,r)),$$

and the *local* $L^q$*-dimension of* $\mu$ *at* $x$ as

$$\dim_q(\mu, x) = \tau_q(\mu, x)/(q-1).$$

Correspondingly we define the *local upper and lower entropy dimensions of* $\mu$ *at* $x \in spt(\mu)$ as

$$\overline{\dim}_1(\mu, x) = \limsup_{r\downarrow 0}\overline{\dim}_1(\mu, B(x,r)),$$
$$\underline{\dim}_1(\mu, x) = \liminf_{r\downarrow 0}\underline{\dim}_1(\mu, B(x,r)).$$

The following theorem explains the choice of notation for the entropy dimensions.

**Theorem 2.2.** [8, Theorem 2.2] *If* $\mu$ *is a measure on a doubling metric space* $X$*, then*

$$\lim_{q\downarrow 1}\dim_q(\mu, x) \le \underline{\dim}_{\mathrm{loc}}(\mu, x) \le \overline{\dim}_{\mathrm{loc}}(\mu, x) \le \lim_{q\uparrow 1}\dim_q(\mu, x), \tag{2.4}$$

*for* $\mu$*-almost all* $x \in X$ *and*

$$\lim_{q\downarrow 1}\dim_q(\mu, x) \le \underline{\dim}_1(\mu, x) \le \overline{\dim}_1(\mu, x) \le \lim_{q\uparrow 1}\dim_q(\mu, x), \tag{2.5}$$

*for every* $x \in \mathrm{spt}(\mu)$.

*Moreover, if the measure has the density point property, then*

$$\underline{\dim}_{\mathrm{loc}}(\mu, x) \le \underline{\dim}_1(\mu, x) \le \overline{\dim}_1(\mu, x) \le \overline{\dim}_{\mathrm{loc}}(\mu, x), \tag{2.6}$$

*for* $\mu$*-almost all* $x \in X$.

The proofs of (2.4) and (2.6) can be found in [9]. However the proof of claim (2.5) is erroneous as it makes use of the incorrect [8, Proposition 3.2]. Our main goal in this article is to give a correct proof for this theorem. A correct proof for claim (2.5) will be given at the end of Section 4.

### 2.3 Counterexample

In [8] the authors aim to formulate an alternate way of defining the $L^q$-spectrum using partitions of the space $X$, which is a little easier to work with than the definition using packings. In this section we introduce the original proposition of [8] which claims that the $L^q$-spectrum of a measure on a subset $A \subset X$ can be calculated using partitions of the space $X$ instead of packings, and provide a counterexample for the proposition.

**Claim.** [8, Proposition 3.2] *If $\mu$ is a measure on a doubling metric space $X$, $A \subset X$ is bounded with $\mu(A) > 0$ and $q \geq 0$, then*

$$\tau_q(\mu, A) = \liminf_{n\to\infty} \frac{\log \sum_{Q\in\mathcal{Q}_n(A)} \mu(Q)^q}{\log \delta_n}.$$

Intuitively, the claim feels plausible and indeed it holds when $A = X$. However there is a problem; in the $\delta$-packing definition of the $L^q$-spectrum we only require the center of the ball to be in the set $A$ and thus the balls of the packing may intersect the complement of $A$ even with small $\delta$ and if the measure in question is concentrated near the set $A$, the original definition may give a much smaller value for the spectrum than the claim above. The following simple counterexample originally proposed by Laurent Dufloux shows that if $A$ is a proper subset of $X$, the proposition above does not necessarily hold.

**Counterexample 2.3.** To simplify the notation we set

$$T_q(\mu, A) = \liminf_{n\to\infty} \frac{\log \sum_{Q\in\mathcal{Q}_n(A)} \mu(Q)^q}{\log \delta_n}.$$

Let $X = [-1, 1]$ with the usual Euclidean metric in $\mathbb{R}$. Let $f : X \to \mathbb{R}$ be defined as

$$f(x) = \begin{cases} 1, & \text{if } x \in [-1, 0], \\ x^{-\frac{1}{2}}, & \text{if } x \in (0, 1]. \end{cases}$$

We define the measure $\mu : \mathcal{L}(X) \to \mathbb{R}^+$ as

$$\mu(A) = \int_A f(x)\mathrm{d}x,$$

where $\mathcal{L}(X)$ is the Lebesgue $\sigma$-algebra on $X$. We show that for $A = [-1, 0]$,

$$\tau_q(\mu, A) < T_q(\mu, A),$$

and thus [8, Proposition 3.2] does not hold.

First we calculate an upper bound for $\tau_q(\mu, A)$. Since $0 \in A$, $\{B(0,\delta)\}$ is a $\delta$-packing of $A$ and

$$S_q(\mu, A, \delta) \geq \mu(B(0,\delta))^q = \left(\int_{-\delta}^{\delta} f(x)\mathrm{d}x\right)^q \geq \left(\int_0^{\delta} x^{-\frac{1}{2}}\mathrm{d}x\right)^q = 2^q\delta^{\frac{1}{2}q}.$$

Hence

$$\tau_q(\mu, A) \leq \liminf_{\delta \downarrow 0} \frac{\log 2^q \delta^{\frac{1}{2}q}}{\log \delta} = \frac{q}{2}.$$

Next we calculate $T_q(\mu, A)$. For each $n \in \mathbb{N}$ we divide the space $X$ in to the dyadic intervals $(m2^{-n}, (m+1)2^{-n}]$ of length $2^{-n}$ and obtain a $2^{-n}$-partition of $X$ denoted by $\mathcal{Q}_n$ (we include the point $-1$ in the appropriate interval). Notice that (2.1) and (2.2) hold for the dyadic partition. Obviously

$$\sum_{Q \in \mathcal{Q}_n(A)} \mu(Q)^q = 2^n 2^{-qn},$$

since the number of dyadic intervals intersecting $A$ is $2^n$. Hence

$$T_q(\mu, A) = \liminf_{n \to \infty} \frac{\log 2^n 2^{-qn}}{\log 2^{-n}} = q - 1 > \tau_q(\mu, A),$$

when $q > 2$.

We note that the mistake in the proof of [8, Proposition 3.2] is in the statement "Since $C_B$ is a cover for $B$" which does not necessarily hold if $A$ is a proper subset of $X$.

# 3 Restricted $L^q$-spectrum and entropy dimension using partitions

## 3.1 Restricted $L^q$-spectrum

As the calculation of $T_q(\mu, A)$ in Counterexample 2.3 shows, the property of [8, Proposition 3.2] is rather desirable, since it would greatly simplify the calculation of the $L^q$-spectrum in some cases. Thus it is in our best interest to try and provide a variant of [8, Proposition 3.2] which holds and allows us to calculate at least the local dimensions, which are the main object of study in this paper, using partitions.

For the rest of the paper we use the notation $\mu_A$ for the restriction of the measure $\mu$ on the set $A \subset X$, i.e. for all $B \subset X$ we set $\mu_A(B) = \mu(B \cap A)$. We call the measure $\mu_A$ the *restricted measure* and the $L^q$-spectrum of the restricted measure the *restricted $L^q$-spectrum*. Similarly we call the entropy and $L^q$-dimensions of the restricted measure the *restricted entropy and $L^q$-dimensions.* Our aim is to provide a formulation of [8, Proposition 3.2] using the restricted $L^q$-spectrum and relate that with the $L^q$-spectrum of the measure on the whole space. First we note that since it is clear that $\mu_X = \mu$, then if $X$ is bounded we have

$$\tau_q(\mu) = \tau_q(\mu_X), \tag{3.1}$$

for all $q \in \mathbb{R}$.

Using the restricted measure we define the *restricted local $L^q$-spectrum of $\mu$ at $x \in spt(\mu)$* as

$$\tau_q^*(\mu, x) = \lim_{r \downarrow 0} \tau_q(\mu_{B(x,r)}, B(x,r)).$$

Similarly we define the *restricted local $L^q$-dimension of $\mu$ at $x \in spt(\mu)$* as

$$\dim_q^*(\mu, x) = \tau_q^*(\mu, x)/(q-1).$$

The *restricted local upper and lower entropy dimensions of $\mu$ at $x \in spt(\mu)$* are defined as

$$\begin{aligned} \overline{\dim}_1^*(\mu, x) &= \limsup_{r \downarrow 0} \overline{\dim}_1(\mu_{B(x,r)}, B(x,r)), \\ \underline{\dim}_1^*(\mu, x) &= \liminf_{r \downarrow 0} \underline{\dim}_1(\mu_{B(x,r)}, B(x,r)), \end{aligned}$$

respectively.

### 3.2 Properties of the $L^q$-spectra and $L^q$-dimensions

First we state a lemma which provides some basic properties for the $L^q$-spectrum and dimension (see [9, Lemma 2.7], for a proof).

**Lemma 3.1.** *If $\mu$ is a measure on a doubling metric space $X$, the set $A \subset X$ is bounded, with $\mu(A) > 0$, setting $q_0 = \inf\{q \in \mathbb{R} : \tau_q(\mu, A) > -\infty\}$, and $s > 0$ as in Lemma 2.1(2 and 3), then*

1. $\tau_1(\mu, A) = 0$,
2. $\min\{0, (q-1)s\} \leq \tau_q(\mu, A) \leq \max\{0, (q-1)s\}$ *for all* $0 \leq q < \infty$,
3. $0 \leq \dim_q(\mu, A) \leq s$ *for all* $0 \leq q < \infty$ *with* $q \neq 1$,
4. *the mapping* $q \mapsto \tau_q(\mu, A)$ *is concave on* $(q_0, \infty)$,
5. *the mapping* $q \mapsto \dim_q(\mu, A)$ *is continuous and decreasing on both* $(q_0, 1)$ *and* $(1, \infty)$.

*Furthermore if $x \in \mathrm{spt}(\mu)$, then all the claims remain true if $\tau_q(\mu, A)$ is replaced by $\tau_q(\mu, x)$ and $\dim_q(\mu, A)$ by $\dim_q(\mu, x)$.*

The following lemma states the basic relationship of the restricted $L^q$-spectrum and the $L^q$-spectrum of the measure on the whole space.

**Lemma 3.2.** *If $\mu$ is a measure on a doubling metric space $X$ and $A \subset X$ is bounded, then*

$$\tau_q(\mu_A, A) \geq \tau_q(\mu, A),$$

*if $q \geq 0$.*

**Proof.** Let $q \geq 0$. Any $\delta$-packing $\mathcal{B}$ of $A \cap \mathrm{spt}(\mu_A)$ is a $\delta$-packing of $A \cap \mathrm{spt}(\mu)$ and since $B \cap A \subset B$, we have

$$\sum_{B \in \mathcal{B}} \mu_A(B)^q \leq \sum_{B \in \mathcal{B}} \mu(B)^q.$$

Therefore

$$S_q(\mu_A, A, \delta) \leq S_q(\mu, A, \delta),$$

and claim follows by taking logarithms, dividing by $\log \delta$ and taking limits. □

The following proposition is the main tool of this paper and relates the restricted local $L^q$-spectrum with the local $L^q$-spectrum of the measure on the whole space and provides us with the useful fact that the local spectra are indeed equal with positive values of $q$.

**Proposition 3.3.** *If $\mu$ is a measure on a doubling metric space $X$ and $q \geq 0$, then*

$$\tau_q^*(\mu, x) = \tau_q(\mu, x), \tag{3.2}$$

*for every $x \in \mathrm{spt}(\mu)$.*

**Proof.** Let $q \geq 0$. By Lemma 3.2 we need only prove that $\tau_q^*(\mu, x) \leq \tau_q(\mu, x)$. Let $x \in \mathrm{spt}(\mu)$, $r > 0$ and $0 < \delta < r$. Let $\mathcal{B}$ be a $\delta$-packing of $B(x, r)$. Then

$$\begin{aligned}\sum_{B \in \mathcal{B}} \mu(B)^q &= \sum_{B \in \mathcal{B}} \mu(B \cap B(x, r + \delta))^q \\ &\leq \sum_{B \in \mathcal{B}} \mu(B \cap B(x, 2r))^q = \sum_{B \in \mathcal{B}} \mu_{B(x,2r)}(B)^q,\end{aligned}$$

and since $\mathcal{B}$ is also a $\delta$-packing of $B(x, 2r)$, we have

$$S_q(\mu, B(x, r), \delta) \leq S_q(\mu_{B(x,2r)}, B(x, 2r), \delta),$$

and the claim then follows by taking logarithms dividing by $\log \delta$ and then taking first $\delta \to 0$ and then $r \to 0$. □

As an immediate consequence we get the following corollary

**Corollary 3.4.** *If $\mu$ is a measure on a doubling metric space $X$ and $1 \neq q \geq 0$, then*

$$\dim_q^*(\mu, x) = \dim_q(\mu, x),$$

*for every $x \in \operatorname{spt}(\mu)$.*

### 3.3 Restricted $L^q$-spectrum using partitions

Next we reformulate [8, Proposition 3.2] using the restricted measure and provide a proof for the statement. Recall that for all $n \in \mathbb{N}$, $\mathcal{Q}_n$ is a $\delta_n$-partition of $X$ for a fixed sequence $(\delta_n)_{n\in\mathbb{N}}$ which satisfies (2.1) and (2.2).

**Proposition 3.5.** *If $\mu$ is a measure on a doubling metric space $X$, $A \subset X$ is bounded with $\mu(A) > 0$ and $q \geq 0$, then*

$$\tau_q(\mu_A, A) = \liminf_{n\to\infty} \frac{\log \sum_{Q\in\mathcal{Q}_n(A)} \mu_A(Q)^q}{\log \delta_n}.$$

**Proof.** The proof closely follows the ideas in proof of [8, Proposition 3.2]. Let $0 < \delta < \delta_1$ and $n \in \mathbb{N}$ so that $\delta_{n+1} < \delta < \delta_n$. First we show that for a constant $c_1(N, \Lambda, q) > 0$, we have

$$S_q(\mu_A, A, \delta) \leq c_1 \left(\frac{\delta_n}{\delta}\right)^s \sum_{Q\in\mathcal{Q}_n(A)} \mu_A(Q)^q, \tag{3.3}$$

where $s = s(N) > 0$ is the constant given by Lemma 2.1(3), $N = N(X)$ is the doubling constant of the metric space $X$ and $\Lambda$ is the constant used in defining the partitions $\mathcal{Q}_n$, which is not dependent on $n$.

Let us fix a $\delta$-packing $\mathcal{B}$ of $A$ and set

$$C_B = \{Q : Q \in \mathcal{Q}_n(A),\, Q \cap B \cap A \neq \emptyset\}$$

for all $B \in \mathcal{B}$. Clearly $\{Q \cap A : Q \in C_B\}$ is a cover for $B \cap A$ and we have

$$\mu_A(B)^q \leq \left(\sum_{Q\in C_B} \mu_A(Q)\right)^q \leq (\#C_B)^q \sum_{Q\in C_B} \mu_A(Q)^q,$$

when $q \geq 0$. Here $\#C_B$ denotes the cardinality of the set $C_B$. By the definition of $\mathcal{Q}_n$, all the sets of $C_B$ are contained in a ball of radius $(1+2\Lambda)\delta_n$, which also by definition has a $\delta_n$-packing of cardinality $\#C_B$. By Lemma 2.1(3) there exists a constant $c_2 = c_2(N, \Lambda) > 0$, such that $\#C_B \leq c_2$ for all $B \in \mathcal{B}$. Hence

$$\sum_{B\in\mathcal{B}} \mu_A(B)^q \leq c_2^q \sum_{B\in\mathcal{B}} \sum_{Q\in C_B} \mu_A(Q)^q.$$

Furthermore, by Lemma 2.1(3) there exists a constant $c_3 = c_3(N,\Lambda) > 0$ so that $\#\{B \in \mathcal{B} : B \cap Q \cap A\} \leq c_3\left(\frac{\delta_n}{\delta}\right)^s$, for all $Q \in Q_n$. Claim (3.3) follows then with $c_1 = c_2^q c_3$.

Finding the estimate for the other direction also requires only a minor alteration to the proof of [8, Proposition 3.2]. For each $Q \in \mathcal{Q}_n(A)$ we choose a point $x_Q \in Q \cap A$ and a ball $B_Q$ such that $Q \subset \Lambda B_Q$ and the collection $\{B_Q : Q \in \mathcal{Q}_n(A)\}$ is a $\delta_n$-packing. Obviously $Q \subset B(x_Q, 2\Lambda\delta_n) \subset 3\Lambda B_Q$, for all $Q \in \mathcal{Q}_n(A)$. Lemma 2.1(4) provides us with a constant $M = M(N,\Lambda) \in \mathbb{N}$ and sets $\mathcal{Q}_1, \ldots, \mathcal{Q}_M$ so that $\mathcal{Q}_n(A) = \bigcup_{i=1}^M \mathcal{Q}_i$ and $\{3\Lambda B_Q : Q \in \mathcal{Q}_i\}$ is a $3\Lambda\delta_n$-packing for all $i \in \{1,\ldots,M\}$ and as a result $\{B(x_Q, 2\Lambda\delta_n) : Q \in \mathcal{Q}_i\}$ is a $2\Lambda\delta_n$-packing with centres in $A$. Clearly we have

$$\bigcup_{Q\in\mathcal{Q}_n(A)} Q \subset \bigcup_{i=1}^{M} \bigcup_{Q\in\mathcal{Q}_i} B(x_Q, 2\Lambda\delta_n).$$

Moreover since $A \cap Q \cap B(x_Q, 2\Lambda\delta_n) \neq \emptyset$ for all $Q \in \mathcal{Q}_n(A)$, we have

$$\bigcup_{Q\in\mathcal{Q}_n(A)} Q \cap A \subset \bigcup_{i=1}^{M} \bigcup_{Q\in\mathcal{Q}_i} B(x_Q, 2\Lambda\delta_n) \cap A,$$

and we may choose $i \in \{1,\ldots,M\}$ so that

$$\sum_{Q\in\mathcal{Q}_n(A)} \mu_A(Q)^q \leq M \sum_{Q\in\mathcal{Q}_i} \mu_A(B(x_Q, 2\Lambda\delta_n))^q \leq M S_q(\mu_A, A, 2\Lambda\delta_n). \tag{3.4}$$

The claim then follows by combining estimates (3.3) and (3.4) and taking logarithms and limits. □

### 3.4 Entropy dimension using partitions

For convenience we present the formulation of $\dim_1(\mu, A)$ for compact sets $A$ using partitions. The proof of the proposition can be found in [8, Proposition 3.4].

**Proposition 3.6.** *If $\mu$ is a measure on a doubling metric space $X$ and $A \subset X$ is compact with $\mu(A) > 0$, then*

$$\overline{\dim}_1(\mu, A) = \limsup_{n\to\infty} \frac{\sum_{Q\in\mathcal{Q}_n(A)} \mu(Q)\log\mu(Q)}{\sum_{Q\in\mathcal{Q}_n(A)} \mu(Q)\log\delta_n},$$
$$\underline{\dim}_1(\mu, A) = \liminf_{n\to\infty} \frac{\sum_{Q\in\mathcal{Q}_n(A)} \mu(Q)\log\mu(Q)}{\sum_{Q\in\mathcal{Q}_n(A)} \mu(Q)\log\delta_n}$$

For the restricted entropy dimensions we obtain a definition using partitions which does not require the set in question to be compact.

**Proposition 3.7.** *If $\mu$ is a measure on a doubling metric space $X$ and $A \subset X$ is bounded with $\mu(A) > 0$, then*

$$\overline{\dim}_1(\mu_A, A) = \limsup_{n\to\infty} \frac{\sum_{Q\in\mathcal{Q}_n(A)} \mu_A(Q)\log\mu_A(Q)}{\mu(A)\log\delta_n},$$
$$\underline{\dim}_1(\mu_A, A) = \liminf_{n\to\infty} \frac{\sum_{Q\in\mathcal{Q}_n(A)} \mu_A(Q)\log\mu_A(Q)}{\mu(A)\log\delta_n}$$

**Proof.** The proof is a simpler version of the proof for [8, Proposition 3.4]. For each $Q \in \mathcal{Q}_n(A)$ we choose a ball $B_Q$ such that $Q \subset \Lambda B_Q$ and $\{B_Q : Q \in \mathcal{Q}_n(A)\}$ is a $\delta_n$-packing. If $Q \in \mathcal{Q}_n(A)$, then for every $y \in Q \cap A$ we have

$$Q \cap A \subset B(y, 2\Lambda\delta_n) \cap A \subset 3\Lambda B_Q \cap A \subset \bigcup_{Q'\in C_Q} Q' \cap A,$$

where $C_Q = \{Q' \in \mathcal{Q}_n(A) : Q' \cap 3\Lambda B_Q \cap A \neq \emptyset\}$. Notice also that

$$A = \bigcup_{Q\in\mathcal{Q}_n(A)} Q \cap A.$$

We then have

$$\begin{aligned}
\sum_{Q\in\mathcal{Q}_n(A)} \mu_A(Q)\log\mu_A(Q) &\leq \sum_{Q\in\mathcal{Q}_n(A)} \int_{Q\cap A} \log\mu_A(B(y, 2\Lambda\delta_n))\mathrm{d}\mu_A(y) \\
&\leq \int_A \log\mu_A(B(y, 2\Lambda\delta_n))\mathrm{d}\mu_A(y) \\
&\leq \sum_{Q\in\mathcal{Q}_n(A)} \mu_A(Q)\log\sum_{Q'\in C_Q} \mu_A(Q').
\end{aligned}$$

By Lemma 2.1(3), there is a constant $c = c(N) < \infty$, such that each $Q'$ is contained in at most $c(3\Lambda)^s$ collections of $C_Q$. Therefore we have

$$\begin{aligned}
&\sum_{Q\in\mathcal{Q}_n(A)} \mu_A(Q) - \sum_{Q\in\mathcal{Q}_n(A)} \mu_A(Q)\log\sum_{Q'\in C_Q} \mu_A(Q') \\
&\qquad = \sum_{Q\in\mathcal{Q}_n(A)} \mu_A(Q)\log\Big(1 + \frac{\sum_{Q'\in C_Q\setminus\{Q\}} \mu_A(Q')}{\mu_A(Q)}\Big) \\
&\qquad \leq \sum_{Q\in\mathcal{Q}_n(A)} \sum_{Q'\in C_Q\setminus\{Q\}} \mu_A(Q') \leq c(3\Lambda)^s\mu(A).
\end{aligned}$$

Combining the previous estimates gives us

$$\begin{aligned}\sum_{Q\in\mathcal{Q}_n(A)} \mu_A(Q)\log\mu_A(Q) &\le \int_A \log\mu_A(B(y,\delta))\mathrm{d}\mu_A(y)\\ &\le \sum_{Q\in\mathcal{Q}_{n-1}(A)} \mu_A(Q)\log\mu_A(Q) + c(3\Lambda)^s\mu(A),\end{aligned}$$

for all $2\Lambda\delta_n \le \delta \le 2\Lambda\delta_{n-1}$. From this and (2.2) the claim follows easily. □

## 4 Relating the dimensions

In this section our aim is to provide relationships between the different notions of dimension discussed in this paper. First we remind ourselves of a small technical lemma introduced in [8].

**Lemma 4.1.** *Suppose $\mu$ is a measure on a doubling metric space $X$ and $A \subset X$ is bounded. Let $s > 0$ and $c > 0$ be as in 2.1(3). Then*

$$\int_A \log\mu(B(y,\delta))\mathrm{d}\mu(y) \ge -\frac{1}{e} - \mu(A)\Big(\log c + s\log\frac{4\mathrm{diam}(A)}{\delta}\Big),$$

*for all $\delta > 0$.*

**Proof.** See [8, Lemma 3.3]. □

Using Propositions 3.6 and 3.7 we get the following result:

**Proposition 4.2.** *If $\mu$ is a measure on a doubling metric space $X$ and $A \subset X$ is compact with $\mu(A) > 0$, then*

$$\begin{aligned}\overline{\dim}_1(\mu_A, A) &= \overline{\dim}_1(\mu, A),\\ \underline{\dim}_1(\mu_A, A) &= \underline{\dim}_1(\mu, A).\end{aligned}$$

**Proof.** First note that for any $y \in A$ and $\delta > 0$ we have $\mu_A(B(y,\delta)) \le \mu(B(y,\delta))$ and hence clearly

$$\begin{aligned}\overline{\dim}_1(\mu_A, A) &= \limsup_{\delta\downarrow 0} \fint_A \frac{\log\mu_A(B(y,\delta))}{\log\delta}\mathrm{d}\mu_A(y)\\ &\ge \limsup_{\delta\downarrow 0} \fint_A \frac{\log\mu(B(y,\delta))}{\log\delta}\mathrm{d}\mu(y) = \overline{\dim}_1(\mu, A),\end{aligned}$$

and the corresponding inequality holds for the lower dimensions as well.

For the other inequality we choose for each $Q \in \mathcal{Q}_n(A)$ a ball $B_Q$ such that $Q \subset \Lambda B_q$ and $\{B_Q : Q \in \mathcal{Q}_n(A)\}$ is a $\delta_n$-packing. If $Q \in \mathcal{Q}_n(A)$, then for every $y \in Q$ we have

$$Q \subset B(y, 2\Lambda\delta_n) \subset 3\Lambda B_Q \subset \bigcup_{Q' \in C_Q} Q',$$

where $C_Q = \{Q' \in \mathcal{Q}_n(A) : Q' \cap 3\Lambda B_Q \neq \emptyset\}$. We set $A_n = \bigcup_{Q \in \mathcal{Q}_n(A)} Q$. Thus

$$\sum_{Q \in \mathcal{Q}_n(A)} \int_Q \log \mu(B(y, 2\Lambda\delta_n)) \mathrm{d}\mu(y) = \int_{A_n} \log \mu(B(y, 2\Lambda\delta_n)) \mathrm{d}\mu(y) \tag{4.1}$$

$$\leq \sum_{Q \in \mathcal{Q}_n(A)} \mu(Q) \log \sum_{Q' \in C_Q} \mu(Q'). \tag{4.2}$$

Since by Lemma 2.1(3) there exists a constant $c_1 = c_1(N) < \infty$, such that $Q' \in \mathcal{Q}_n(A)$ is contained in at most $c_1(3\Lambda)^s$ collections $C_Q$, we have

$$\begin{aligned}
&\sum_{Q \in \mathcal{Q}_n(A)} \mu(Q) \log \sum_{Q' \in C_Q} \mu(Q') - \sum_{Q \in \mathcal{Q}_n(A)} \mu(Q) \log \mu(Q) \\
&\qquad = \sum_{Q \in \mathcal{Q}_n(A)} \mu(Q) \log \left( 1 + \frac{\sum_{Q' \in C_Q \setminus \{Q\}} \mu(Q')}{\mu(Q)} \right) \\
&\qquad \leq \sum_{Q \in \mathcal{Q}_n(A)} \sum_{Q' \in C_Q \setminus \{Q\}} \mu(Q') \leq c_1(3\Lambda)^s \mu(B_0),
\end{aligned}$$

where $B_0$ is a ball centered at A with radius $\operatorname{diam}(A) + 2\Lambda\delta_n$. Combining this with (4.1) we get

$$\int_{A_n} \log \mu(B(y, 2\Lambda\delta_n)) \mathrm{d}\mu(y) \leq \sum_{Q \in \mathcal{Q}_n(A)} \mu(Q) \log \mu(Q) + c_1(3\Lambda)^s \mu(B_0). \tag{4.3}$$

Next we note that $\mu(Q) = \mu_{A_n}(Q)$, since $Q \subset A_n$ for any $Q \in \mathcal{Q}_n(A)$ and therefore $\mu(Q) = \mu_A(Q) + \mu_{A_n \setminus A}(Q)$ for any $Q \in \mathcal{Q}_n(A)$. Thus

$$\begin{aligned}
\sum_{Q \in \mathcal{Q}_n(A)} \mu(Q) \log \mu(Q) &= \sum_{Q \in \mathcal{Q}_n(A)} \mu_A(Q) \log \mu(Q) + \sum_{Q \in \mathcal{Q}_n(A)} \mu_{A_n \setminus A}(Q) \log \mu(Q) \\
&\leq \sum_{Q \in \mathcal{Q}_n(A)} \mu_A(Q) \log \mu(Q) + \log \mu(A_n) \sum_{Q \in \mathcal{Q}_n(A)} \mu_{A_n \setminus A}(Q) \qquad (4.4) \\
&= \sum_{Q \in \mathcal{Q}_n(A)} \mu_A(Q) \log \mu(Q) + \log \mu(A_n) \mu(A_n \setminus A).
\end{aligned}$$

We also have

$$\sum_{Q\in\mathcal{Q}_n(A)}\mu_A(Q)\log\mu(Q)-\sum_{Q\in\mathcal{Q}_n(A)}\mu_A(Q)\log\mu_A(Q)$$
$$=\sum_{Q\in\mathcal{Q}_n(A)}\mu_A(Q)\log\frac{\mu(Q)}{\mu_A(Q)}=\sum_{Q\in\mathcal{Q}_n(A)}\mu_A(Q)\log\frac{\mu_A(Q)+\mu_{A_n\setminus A}(Q)}{\mu_A(Q)} \tag{4.5}$$
$$=\sum_{Q\in\mathcal{Q}_n(A)}\mu_A(Q)\log\left(1+\frac{\mu_{A_n\setminus A}(Q)}{\mu_A(Q)}\right)\leq\sum_{\mathcal{Q}_n(A)}\mu_{A_n\setminus A}(Q)=\mu(A_n\setminus A)$$

and by combining (4.3), (4.4) and (4.5) we get

$$\int_{A_n}\log\mu(B(y,\delta))\mathrm{d}\mu(y)\leq\sum_{Q\in\mathcal{Q}_n(A)}\mu_A(Q)\log\mu_A(Q) \tag{4.6}$$
$$+\mu(A_n\setminus A)(1+\log\mu(A_n))+c_1(3\Lambda)^s\mu(B_0),$$

for all $0<\delta\leq 2\Lambda\delta_n$. Since $A$ is compact we have $\lim_{n\to\infty}\mu(A_n\setminus A)=0$ and by Lemma 4.1,

$$\lim_{n\to\infty}\frac{1}{\log\delta_n}\int_{A_n\setminus A}\log\mu(B(y,2\Lambda\delta_n))\mathrm{d}\mu(y)=0.$$

From this, (4.6) and Proposition 3.7 the claim follows. □

**Corollary 4.3.** *If $\mu$ is a measure on a doubling metric space $X$, then*

$$\overline{\dim}_1^*(\mu,x)=\overline{\dim}_1(\mu,x),$$
$$\underline{\dim}_1^*(\mu,x)=\underline{\dim}_1(\mu,x).$$

**Proof.** Apply Proposition 4.2 to the compact balls $B(x,r)$ and take the limit. □

Next we show that the definitions of restricted entropy dimensions and $L^q$-dimensions are consistent with the monotonicity of the $L^q$-dimensions. The result resembles [8, Proposition 3.7], the proof of which is slightly incorrect since it makes use of [8, Proposition 3.2]. Our proof follows the ideas of the proof of [8, Proposition 3.7], but we use Proposition 3.5 instead of [8, Proposition 3.2].

**Proposition 4.4.** *If $\mu$ is a measure on a doubling metric space $X$ and $A\subset X$ is bounded with $\mu(A)>0$, then*

$$\lim_{q\downarrow 1}\dim_q(\mu_A,A)\leq\underline{\dim}_1(\mu_A,A)\leq\overline{\dim}_1(\mu_A,A)\leq\lim_{q\uparrow 1}\dim_q(\mu_A,A).$$

**Proof.** First notice that the existence of the limits follows from Lemma 3.1(5). The claim then follows if we can show that

$$\tau_q(\mu_A, A)/(q-1) \geq \overline{\dim}_1(\mu_A, A) \geq \underline{\dim}_1(\mu_A, A) \geq \tau_p(\mu_A, A)/(p-1),$$

for $0 < q < 1 < p$. We define a function $h_n(q) = \log \sum_{Q \in \mathcal{Q}_n(A)} \mu_A(Q)^q$, for all $q \geq 0$. First we show that $h_n(q)$ is convex. Take $q, p \in \mathbb{R}$ and $t \in [0,1]$. Now

$$\begin{aligned} h_n(tq + (1-t)p) &= \log \sum_{Q \in \mathcal{Q}_n(A)} \mu_A(Q)^{tq} \mu_A(Q)^{(1-t)p} \\ &\leq \log \Big( \sum_{Q \in \mathcal{Q}_n(A)} \mu_A(Q)^q \Big)^t \Big( \sum_{Q \in \mathcal{Q}_n(A)} \mu_A(Q)^p \Big)^{1-t} \\ &= t h_n(q) + (1-t) h_n(p). \end{aligned}$$

by Hölder's inequality. Note that $Q_n(A)$ has only a finite number of elements, which implies that $h_n(q)$ is differentiable, and differentiating gives us

$$\begin{aligned} h_n'(1) &= \frac{\sum_{Q \in \mathcal{Q}_n(A)} \mu_A(Q) \log \mu_A(Q)}{\sum_{Q \in \mathcal{Q}_n(A)} \mu_A(Q)} \\ &= \frac{\sum_{Q \in \mathcal{Q}_n(A)} \mu_A(Q) \log \mu_A(Q)}{\mu(A)}. \end{aligned}$$

By the convexity of $h_n(q)$ we then have

$$\frac{h_n(q) - h_n(1)}{q-1} \leq h_n'(1) \leq \frac{h_n(p) - h_n(1)}{p-1}.$$

Using the above estimates we calculate

$$\begin{aligned} \frac{1}{q-1} \liminf_{n\to\infty} \frac{\log \sum_{Q \in \mathcal{Q}_n(A)} \mu_A(Q)^q}{\log \delta_n} &= \limsup_{n\to\infty} \frac{h_n(q) - h_n(1)}{(q-1)\log \delta_n} \\ &\geq \limsup_{n\to\infty} \frac{\log \sum_{Q \in \mathcal{Q}_n(A)} \mu_A(Q) \log \mu_A(Q)}{\mu(A) \log \delta_n} \\ &\geq \liminf_{n\to\infty} \frac{\log \sum_{Q \in \mathcal{Q}_n(A)} \mu_A(Q) \log \mu_A(Q)}{\mu(A) \log \delta_n} \\ &\geq \liminf_{n\to\infty} \frac{h_n(p) - h_n(1)}{(p-1)\log \delta_n} = \frac{1}{p-1} \liminf_{n\to\infty} \frac{\log \sum_{Q \in \mathcal{Q}_n(A)} \mu_A(Q)^p}{\log \delta_n}, \end{aligned}$$

where the first and last equalities hold since $h_n(1)$ does not depend on $n$. Now the result follows from Propositions 3.5 and 3.7. ∎

*Remark* 4.5. Proposition 4.4 does not provide an immediate proof for [8, Proposition 3.7], since we only have $\lim_{q\downarrow 1} \dim_q(\mu_A, A) \leq \lim_{q\downarrow 1} \dim_q(\mu, A)$ and $\lim_{q\uparrow 1} \dim_q(\mu_A, A) \geq \lim_{q\uparrow 1} \dim_q(\mu, A)$, by Lemma 3.2. The statement of Proposition 4.4 is enough to show that the definitions of local entropy dimensions are consistent with the monotonicity of the local $L^q$ dimensions, and since our focus is in the local case, [8, Proposition 3.7] remains open. Notice that our proposition does provide the estimates for the whole space i.e. Proposition 4.4 together with (3.1) implies that $\lim_{q\downarrow 1} \dim_q(\mu) \leq \underline{\dim}_1(\mu) \leq \overline{\dim}_1(\mu) \leq \lim_{q\uparrow 1} \dim_q(\mu)$.

As the main result of this paper we provide a correct proof for [8, Theorem 2.2]. A different proof which does not utilize the same global methods used here can be found in [7], which is an earlier arXiv preprint of [8, 9].

**Proof. (Of claim (2.5) in the statement of Theorem 2.2.)**
By Lemma 3.1(5) the limits exist and by Proposition 4.4 we have

$$\begin{aligned} &\dim_p(\mu_{B(x,r)}, B(x,r)) \leq \underline{\dim}_1(\mu_{B(x,r)}, B(x,r)) \\ &\leq \overline{\dim}_1(\mu_{B(x,r)}, B(x,r)) \leq \dim_q(\mu_{B(x,r)}, B(x,r)), \end{aligned}$$

for $0 < q < 1 < p$, and for every $x \in \operatorname{spt}(\mu)$ and $r > 0$. By taking $r \downarrow 0$ we get

$$\dim_p^*(\mu, x) \leq \underline{\dim}_1^*(\mu, x) \leq \overline{\dim}_1^*(\mu, x) \leq \dim_q^*(\mu, x),$$

and the claim follows from Corollaries 3.4 and 4.3. □

## 5 Discussion

The focus of [8] is in establishing the theory for local multifractal analysis and this paper shows that the local theory in [8] is correct despite the fact that multiple proofs make use of a slightly incorrect result. Although we fail to provide similar results to the second claim of Theorem 2.2 for the case where $A \subsetneq X$ (which was attempted in [8]), our methods do establish the results when considering the two arguably most important cases, the whole space $X$ and the local case. To complete the paper we give a remark considering the applications of the local properties discussed in the paper.

As application of the theory in [8] the authors provide a local multifractal formalism for Moran constructions in doubling metric spaces and use their faulty Proposition 3.2 in the proof of Theorem 4.2. We note here that with only trivial modifications to their proof one can use our Proposition 3.5 in finding the value of $\tau_q^*(\mu, x)$ and then use Proposition 3.3 to obtain a correct proof for the theorem.

## Acknowledgments

I want to express my gratitude to Ville Suomala for insightful discussions and invaluable help in writing the paper and Laurent Dufloux for noticing the mistake in [8] and constructing the counterexample.

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**Received**: October 13, 2020 **Accepted**: November 2, 2020
**Communicated by Tamás Forgács**