

Optimal Tilings of Bipartite Graphs Using Self-Assembling DNA

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Abstract - Motivated by the recent advancements in nanotechnology and the discovery of new laboratory techniques using the Watson-Crick complementary properties of DNA strands, formal graph theory has recently become useful in the study of self-assembling DNA complexes. Construction methods based on graph theory have resulted in significantly increased efficiency. We present the results of applying graph theoretical and linear algebra techniques for constructing crossed-prism graphs, crown graphs, book graphs, stacked book graphs, and helm graphs, along with kite, cricket, and moth graphs. In particular, we explore various design strategies for these graph families in two sets of laboratory constraints.

Keywords : graph theory; self-assembling DNA; tiling; nanotechnology; bipartite graphs

Mathematics Subject Classification (2020) : 05C90

1 Introduction

Recent advancements in micro-biology and nanotechnology have motivated many fields of research including those outside of the practical laboratory environment. One such field, and that which this paper pertains to, is graph theory. In the setting of this research, we seek to build nanostructures out of strands of DNA while representing these structures graphically. As micro-biology and nanotechnology continue to advance, the structures they develop continually get smaller and smaller and it can become difficult to produce the target structure. To resolve this, laboratories are studying ways to create nano-parts from synthetic DNA which utilize the Watson-Crick complementary properties of DNA strands to self-assemble into the targeted nanostructure [9, 15]. This provides motivation for the application of graph theory in the area of research since many of the nanostructures exhibit graphical characteristics. The hope is to design self-assembling DNA structures that, when mixed together, can create the graphs in question. Applications for this are continuously being developed and include targeted drug delivery, DNA splicing, and more (see [1, 6, 5, 7, 10, 11, 12, 13, 14, 21, 22]). The goal of this research is to design nanostructures, efficiently using the minimum number of molecules.



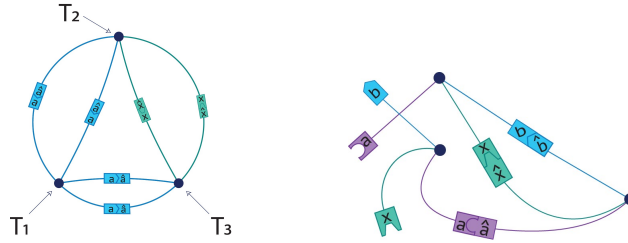


Figure 1: Complete vs Incomplete Complex

2 Research Methods

In this research we will use the flexible-tile model as described in [3] and use the graph theoretical formalism of [4] and [8] to help build targeted structures from self-assembling DNA. We define a **tile** to be a vertex with n -half edges, where n is the degree of the vertex that the tile is representing. The half edges of this tile represent one half of a strand of DNA that wants to bond to its complementary half. For the purposes of generality, we label these “sticky-ends” or “cohesive-ends” of partial single strands of DNA, with a letter, such as a , and their complement with a hatted letter, such as \hat{a} . When complementary cohesive-ends bond together we refer to this as a **bond-edge**, and we call the letter labels **bond-edge types**. We also proceed under the assumption that our DNA-arms are flexible, and can bend in any way to also form our target structures. Thus, we can represent a tile as a set of sticky ends, such as the degree-3 tile $\{a, a, \hat{a}\}$, which can also be denoted as $\{a^2, \hat{a}\}$ or $\{a^2\hat{a}\}$. We define a **pot**, P , as a set of tiles that can realize, or create, a graph. Theoretically, we assume that once a tile type is made, a pot with that tile type can have “infinitely many” of those tile types. Once we have a pot, those tiles are able to self-assemble into complexes either complete or incomplete as seen in Figure 1. Labs generally want to create complete complexes. The formal definition for tiles and pots are repeated here for the convenience of the reader.

Definition 2.1 A ***k -armed branched junction molecule*** is a molecule whose arms are formed from strands of DNA, possibly multiple strands. At the end of each of these arms is a region of unsatisfied bases, forming a cohesive-end. Arms with complementary cohesive-ends can bond via Watson-Crick base pairing.

Definition 2.2 A ***tile*** t_n is a vertex of degree k with k incident half-edges which we use to mathematically represent a k -armed branched junction molecule. Examples of these can be seen in Figure 2.

Definition 2.3 A ***pot*** is a set of tiles such that if a cohesive-end type appears in the multi-set of any tile in the pot, its complement also appears on some tile in the pot. A ***pot*** P with tiles t_1, \dots, t_n is written $P = \{t_1, \dots, t_n\}$.

Definition 2.4 We say a pot P **realizes** a graph G if the collection of tiles constructs the same structure as G .



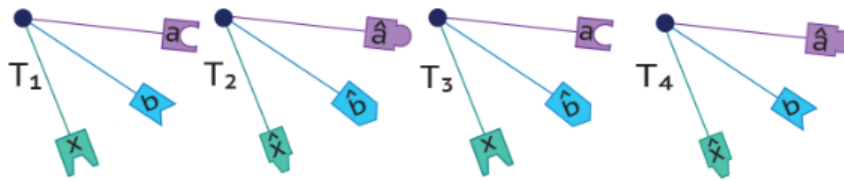


Figure 2: A pot containing 4 tiles.

Scenario	Can Create Smaller Order Graphs?	Can Create Non-isomorphic Graphs?
1	Yes	Yes
2	No	Yes
3	No	No

Table 1: Table summarizing Scenarios 1, 2, and 3

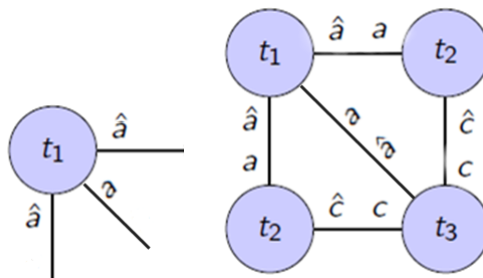


Figure 3: An example of a tile (left) and a realized graph from pot $\{t_1, t_2, t_3\}$ (right).

Figure 3 displays a graph which is realized by a pot of 3 tiles. The tiles from the pot that realized this graph are $t_1 = \{a, \hat{a}^2\}$, $t_2 = \{a, \hat{c}\}$, and $t_3 = \{\hat{a}, c^2\}$. In order to realize this graph, we used one t_1 , two t_2 , and one t_3 . The goal of this research is to produce pots which realize graphs in as few bond types and as few tile types as possible.

There are three laboratory constraints under which these pots are produced, referred to as Scenarios 1, 2, and 3. Under Scenario 1, a pot of tiles can realize a graph of smaller order than the target graph and not isomorphic to our target graph. Under Scenario 2, the pot can not realize a graph of smaller order than that of the target graph, but non-isomorphic graphs of the same order are still acceptable. Scenario 3 dictates that the pot cannot realize any smaller ordered graph and cannot realize any non-isomorphic graph of the same order as our target graph. Table 1 summarizes these restrictions. We denote the number of tile types and bond-edge types needed to tile graph G in Scenario m as $T_m(G)$ and $B_m(G)$ respectively.

This particular research focuses on constructing crossed-prism graphs, crown graphs, book graphs, stacked book graphs, and helm graphs, along with kite, cricket, and moth graphs. When we refer to a target graph, we will be generally speaking about a member of an arbitrary size of one of these families (except for kite, cricket, and moth of which



there is only one graph type).

3 Preexisting Results

We will begin by reviewing some common graph theory notation and definitions used in this research followed by a series of useful results from [5].

3.1 Graph Theory Notation and Definitions

For the purpose of our research, we will use standard graph theory definitions. We will define a graph G to be a set of vertices $V(G)$ and a set of edges $E(G)$. We say $a \in V(G)$ and $b \in V(G)$ are adjacent if and only if the $\{a, b\} \in E(G)$. Each vertex has a valency or degree associated to it, which is equal to the number of edges which are connected to it. The valency sequence of a graph is then the set of valencies or degrees, of each vertex in G . We denote the number of distinct degrees in the valency sequence of G as $av(G)$. $ov(G)$ and $ev(G)$ denote the number of distinct odd and even degrees in the valency sequence of G respectively. We provide an example of a kite graph with a tail of length 1, pictured in Figure 4. The $av(G) = |\{1, 2, 3\}| = 3$, the $ev(G) = |\{2\}| = 1$, and the $ov(G) = |\{1, 3\}| = 2$.

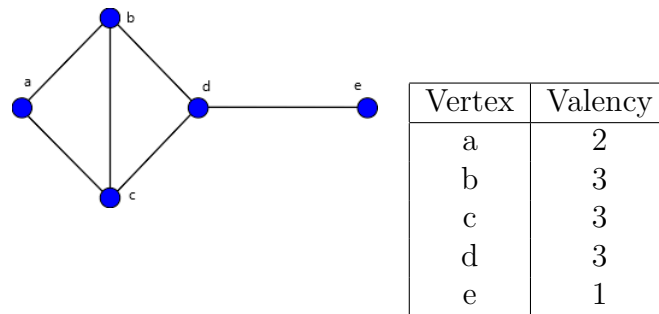


Figure 4: A kite graph and a table containing the valency of each vertex

3.2 Scenario 1

Much is already known about how graphs are tiled in Scenario 1. Theorems 3.1 and 3.2 from [5] establish a relation from the degree sequence of any graph G to its tiling, relying on the size of the set of all valencies av of G and the sizes of the even valency set ev and the odd valency set ov of G . We also know from [5] and Lemma 1 from [2], that all graphs can be tiled under Scenario 1 using 1 bond-edge type.

Theorem 3.1 *For any graph G , $av(G) \leq T_1(G) \leq ev(G) + 2ov(G)$.*

Theorem 3.2 *For any graph G , $B_1(G) = 1$.*



Another useful theorem from [5] is a result regarding k -regular graphs. A **k -regular graph** is a graph in which all vertices of the graph have exactly degree k .

Theorem 3.3 *If G is a k -regular graph, $T_1(G)$ is 1 if k is even and 2 if k is odd.*

3.3 Scenario 2

The most relevant general Scenario 2 result we use in this research is the following theorem from [5], which relates the number of necessary bond-edge types to the number of tiles in Scenario 2.

Theorem 3.4 *Given a graph G , $B_2(G) + 1 \leq T_2(G)$.*

As described in [5], the proportions of the tiles necessary to represent a graph G can be expressed as a system of equations. Using these equations we can check the viability of our pots in Scenario 2. However, this method is only able to provide information on whether or not the specified tiles can successfully realize a graph of the target order or smaller; it does not reveal if the target graph can be realized in fewer tiles. The system of equations can be represented by something called the construction matrix which is defined below.

Definition 3.5 *Let $P = \{t_1, \dots, t_p\}$ be a pot and let $z_{i,j}$ denote the net number of cohesive-ends of type a_i on tile t_j , where un-hatted cohesive-ends are counted positively and hatted cohesive-ends are counted negatively. Then the following system of equations must be satisfied by any graph in any complete complex :*

$$\begin{aligned} z_{1,1}r_1 + z_{1,2}r_2 + \dots + z_{1,p}r_p &= 0 \\ &\vdots \\ z_{m,1}r_1 + z_{m,2}r_2 + \dots + z_{m,p}r_p &= 0 \\ r_1 + r_2 + \dots + r_p &= 1 \end{aligned}$$

The **construction matrix** of P , denoted $M(P)$, is the corresponding augmented matrix:

$$M(P) = \left(\begin{array}{cccc|c} z_{1,1} & z_{1,2} & z_{1,3} & \dots & z_{1,n} & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ z_{m,1} & z_{m,2} & z_{m,3} & \dots & z_{m,n} & 0 \\ \hline 1 & 1 & 1 & \dots & 1 & 1 \end{array} \right)$$

The construction matrix is formulated by allocating each tile to a separate column and each letter used, such as a or b , to a different row. The unhatted letters are assigned a value of 1, while the hatted letters are assigned a value of -1 . In each entry of the matrix, the numerical values are calculated based on the sum of the aforementioned weights assigned to the letters used for the particular tile. Thus, for example, the entry $z_{1,1}$ (the entry in the first row and first column) is equal to the sum of the a 's and \hat{a} 's on the first tile. In



Figure 5, the value $z_{1,2}$ is -1 as the second tile in the pots which this matrix represents must have 2 \hat{a} 's and 1 a , because it is the first row and $-2 + 1 = -1$.

We use the construction matrix to solve a system of linear equations where all of the equations except for the last equation gives an equation for the net total hatted and unhatted letters for each graph. Recall these letters represent the bond-edge types. In a realized graph, the number of hatted cohesive-end types must equal the number of unhatted cohesive-end types, so each equation should equal zero (save the last row). Since each column of the matrix represents the sum of the unhatted and hatted letters of that bond type in the tile, the last equation is all 1's because the sum of the proportions of each tile type should add up to 100%.

The construction matrix is used primarily to determine whether a given pot satisfies the restrictions of Scenario 2 for a desired target graph. When $M(P)$ has a unique solution $\langle r_1, \dots, r_p \rangle$, the smallest order of a graph that can be constructed from the given pot of tiles is the least common denominator of the r_i 's. [5] gives the following result relating to the construction matrix.

Proposition 3.6 *Let $P = \{t_1, \dots, t_p\}$ be a pot. Then:*

1. *If a graph G of order n is realized by P using R_j tiles of type t_j , then $\frac{1}{n}\langle R_1, \dots, R_p \rangle$ is a solution of the construction matrix $M(P)$.*
2. *If $\langle r_1, \dots, r_p \rangle$ is a solution of the construction matrix $M(P)$, and n is a positive integer such that $nr_j \in \mathbb{Z}_{\geq 0}$ for all j , then there is a graph of order n that is realized by using nr_j tiles of type t_j .*
3. *The smallest order of graphs realized by a pot P is given by $m_P = \min\{\text{lcm}\{b_j | r_j \neq 0 \text{ and } r_j = a_j/b_j\}\}$, where $\langle r_1, \dots, r_p \rangle$ is in the solution space of the construction matrix of a pot P , and where the minimum is taken over all solutions to $M(P)$ such that $r_j \geq 0$ and a_j/b_j is in reduced form for all j .*

Sometimes, the construction matrix yields infinitely many solutions. Given the specific requirements of the assembly process and special form of the construction matrix, [2] outlines a Maple program which solves the Integer Linear Programming problem in the case of only a few degrees of freedom. This program is used in some of the results of this paper.

Figure 5 shows the construction matrix in its reduced form for the tiling of the complete graph on four vertices, denoted K_4 . The solution confirms that the smallest graph which can be produced by this pot is a graph of order four.

$$M(K_4) = \left(\begin{array}{cc|c} 3 & -1 & 0 \\ 1 & 1 & 1 \end{array} \right) \quad M(K_4) = \left(\begin{array}{cc|c} 1 & 0 & \frac{1}{4} \\ 0 & 1 & \frac{3}{4} \end{array} \right)$$

Figure 5: The construction matrix of K_4 which can be tiled with the pot, $P = \{t_1 = \{a^3\}, t_2 = \{a, \hat{a}^2\}\}$ and its row reduced form (right)



As can be seen in the matrices, K_4 can be tiled using just two tiles in Scenario 2, $t_1 = a^3$, denoted by the entry labeled 3 in the $(1, 1)$ entry of the matrix on the left, and $t_2 = \{a, \hat{a}^2\}$ as denoted by the -1 in the $(1, 2)$ spot in the matrix on the left. After row reduction, we find that our pot can produce a graph with 4 vertices and not any graph of smaller order, as seen in the denominators of the right hand matrix, requiring one t_1 and three t_2 as seen in the value of the numerators.

4 Results

4.1 Book Graph Results

We start with our results from the book graph family. This family gets its name from the way its member graphs can be visualized in three dimensional space. Formally defined below, we will use some colloquial terms to assist in talking about the parts of the graph. In the book graphs, the vertices which have a degree greater than two are the “top” and “bottom” of the “spine.” The degree-2 vertices which are attached to these vertices form the “pages” of which there are m of in a book graph BK_m . Note that sometimes book graphs are denoted using B_m , but we are using BK_M since we use the notation B_n to denote the minimum number of bond-edge types for a graph in Scenario n .

Definition 4.1 *The m -book graph, denoted BK_m , is defined as the graph Cartesian product (denoted by \square) $BK_m = S_m \square P_2$ where S_m is a star graph and P_2 is the path graph on two vertices. Members of this family are seen in Figure 6.*

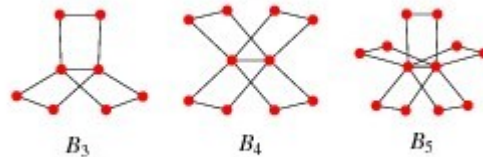


Figure 6: Book graph family (pictures from [16]), note B_m notation is used instead of BK_m notation.

The results for Scenario 1 in this family follow directly from Theorem 3.1 and Theorem 3.2.

Theorem 4.2 *Given a book graph BK_m , $B_1(BK_m) = 1$ and $T_1(BK_m) = 2$ if m is odd and $T_1(BK_m) = 3$ if m is even.*

Proof. It follows from Theorem 3.2 that $B_1(BK_m) = 1$.

Case 1: Suppose BK_m is a book graph where m is odd. Note that $av(BK_m) = |\{2, m + 1\}| = 2$. The following pot realizes BK_m : $P = \{t_1 = \{a^{\frac{m+1}{2}}, \hat{a}^{\frac{m+1}{2}}\}, t_2 = \{a, \hat{a}\}\}$. P is therefore the minimum sized pot which realizes G in Scenario 1 by Theorem 3.1.



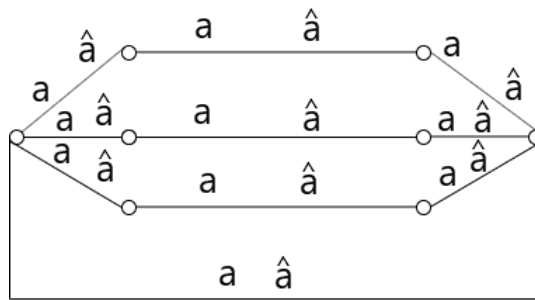


Figure 7: BK_3 tiled under Scenario 1

Case 2: Suppose BK_m is a book graph where m is even. Note that $av(BK_m) = |\{2, m+1\}| = 2$. Assume $T_1(BK_m) = 2$. This would imply that both of the vertices of BK_m with degree- $(m+1)$ would need the same tile. Furthermore, we need the following equation to be satisfied: $2n + 2mk = 0$ where $n = \{\pm 1, \pm 3, \dots, \pm(m+1)\}$ and $k = \{0, \pm 2\}$ (the possible net a 's for the tile of degree $m+1$ and 2 respectively). If $k = 0$, since n is odd, no possible value for n will satisfy the equation. If $k = \pm 2$, $n = \mp 2m$ which is not possible since n is odd and m is even. Thus, $T_1(G) \geq 3$. The following pot $P = \{t_1 = \{a^{m+1}\}, t_2 = \{\hat{a}^{m+1}\}, t_3 = \{a, \hat{a}\}\}$ realizes G and therefore $T_1(BK_m) = 3$. \square

Figure 7 shows a book graph drawn in its planar form tiled under Scenario 1.

Lemma 4.3 Given a book graph BK_m , $B_2(BK_m) \geq 2$ for all m .

Proof. We proceed by contradiction. Assume $B_2(BK_m) = 1$. Since the degree-2 vertices are adjacent, if we want to repeat the tile for each degree-2 vertex, each tile should have one a and one \hat{a} . Unfortunately, this makes a graph of one vertex. If we allow for two different tile types for our degree-2 vertices, then we must use $t_1 = \{a^2\}$ and $t_2 = \{\hat{a}^2\}$ which will create a graph of order two. Thus $B_2(BK_m) > 1$. \square

Theorem 4.4 Given book graph BK_m , where m is even, $B_2(BK_m) = 2$, $T_2(BK_m) = 3$.

Proof. From Lemma 4.3, $B_2(BK_m) \geq 2$. The following pot constructs BK_m when m is even. $P = \{t_1 = \{a^2\}, t_2 = \{\hat{a}, b\}, t_3 = \{\hat{a}^m, b\}, t_4 = \{\hat{b}^{m+1}\}\}$. The general construction matrix for BK_m when m is even is

$$M(P) = \left(\begin{array}{cccc|c} 2 & -1 & -m & 0 & 0 \\ 0 & 1 & 1 & -(m+1) & 0 \\ 1 & 1 & 1 & 1 & 1 \end{array} \right). \quad (1)$$

Using a Maple program from [2] and given the restrictions on the free variable within this application of the construction matrix, we found the solution set of this pot to be $\{(\frac{m}{2(m+1)}, \frac{m}{2(m+1)}, \frac{1}{2(m+1)}, \frac{1}{2(m+1)})\}$. Thus, $B_2(BK_m) = 2$ and $T_2(BK_m) = 3$ for BK_m and m is even. Note the number of tiles is minimal from Theorem 3.4. \square



Theorem 4.5 Given book graph BK_m , where m is odd, $B_2(BK_m) > 2$.

Proof. Suppose BK_m is a book graph where m is odd. Recall that the distinct degrees in the degree sequence of BK_m are $\{2, m + 1\}$. We proceed by contradiction. Suppose $B_2(BK_m) = 2$. As was the case in Lemma 4.3, we know that the degree-2 vertices require two different tiles which have at least two different bond-edge types. Thus, without loss of generality, we have the following cases for our two degree-2 tiles.

Case 1: Suppose, without loss of generality, our pot includes $t_1 = \{a, b\}$ and $t_2 = \{a, \hat{b}\}$. Since the degree- $(m + 1)$ vertices are adjacent and each degree-2 vertex is adjacent to another degree-2 vertex (through the b and \hat{b} ends) we cannot use the same tile for the degree- $(m + 1)$ vertices. In this case, they must be without loss of generality either $t_3 = \{\hat{a}^m, b\}$ and $t_4 = \{\hat{a}^m, \hat{b}\}$ or $t_3 = \{a, \hat{a}^m\}$ and $t_4 = \{\hat{a}^{m+1}\}$. Both cases create smaller graphs. The former can create a graph of order $(m + 1)$ using one t_3 , $\frac{m-1}{2}$ number of t_1 s, and $\frac{m+1}{2}$ number of t_2 (among other smaller graphs). The latter can construct a graph of order m using one t_3 and $\frac{m-1}{2}$ number of t_1 tiles and $\frac{m-1}{2}$ number of t_2 tiles (among other smaller graphs).

Case 2: Suppose our pot includes, without loss of generality, $t_1 = \{a, b\}$ and $t_2 = \{\hat{a}, \hat{b}\}$. This forms a graph of order two and thus these tile types cannot be in our pot.

Case 3: Suppose our pot includes, without loss of generality, $t_1 = \{a^2\}$. Since the degree-2 vertices are adjacent, this means our second two branched tile must have exactly one \hat{a} . We can't have two \hat{a} 's since this would cause a graph of order two. Thus without loss of generality, suppose we have $t_2 = \{\hat{a}, b\}$. In total we have eight possible two branch tiles which can be considered in Scenario 2 (since $\{a, \hat{a}\}$ is not a valid option): $\{a^2\}$, $\{b^2\}$, $\{a, b\}$, $\{a, \hat{b}\}$, $\{b, \hat{a}\}$, $\{\hat{b}, \hat{a}\}$, $\{\hat{a}^2\}$, and $\{\hat{b}^2\}$. If our graph is using $t_1 = \{a^2\}$ and $t_2 = \{\hat{a}, b\}$, we can't use $\{\hat{b}, \hat{a}\}$ since this tile along with $\{a^2\}$ and $\{\hat{a}, b\}$ will create a graph of order three. We can't use $\{\hat{b}^2\}$ since that tile along with $\{a^2\}$ and $\{\hat{a}, b\}$ will create a graph of order four. We can't use $\{b^2\}$ because we would need a degree-2 vertex with \hat{b} because our degree-2 vertices are adjacent. Thus the only possible additional option we can use for an additional degree-2 vertex is $\{a, b\}$.

First we will assume our pot does not use $\{a, b\}$ and only uses two types of tiles for the degree-2 vertices: $t_1 = \{a^2\}$ and $t_2 = \{\hat{a}, b\}$. We now have the following cases for our degree- $(m + 1)$ vertices: $t_3 = \{\hat{a}^m, *\}$ and $t_4 = \{\hat{b}^m, \hat{*}\}$ or $t_3 = \{\hat{a}^k, \hat{b}^j, *\}$ and $t_4 = \{\hat{a}^j \hat{b}^k, \hat{*}\}$ where $k + j = m$.

Case 3a: Suppose $t_3 = \{\hat{a}^m, *\}$ and $t_4 = \{\hat{b}^m, \hat{*}\}$. We now goes through the cases for $*$ and show that we will always be able to construct a smaller graph.

- If $t_3 = \{\hat{a}^{m+1}\}$, then $t_4 = \{a, \hat{b}^m\}$ and several smaller graphs can be made. One such example is a graph of order $(\frac{3m+1}{2})$ with $(\frac{m-1}{2})$ number of $\{a^2\}$ tiles, m number of $\{\hat{a}, b\}$ tiles, and 1 t_4 tile.
- If $t_3 = \{\hat{a}^m, a\}$, then $t_4 = \{\hat{a}, \hat{b}^m\}$ and several smaller graphs can be made. One such example is a graph of order $(2(m + 1) - \frac{m+1}{2})$ with $(\frac{m+1}{2})$ number of $\{a^2\}$ tiles, m number of $\{\hat{a}, b\}$ tiles, and one t_4 tile.



- If $t_3 = \{\hat{a}^m, b\}$, then $t_4 = \{\hat{b}^{m+1}\}$ and t_4 can connect with $(m + 1)$ number of $\{\hat{a}, b\}$ tiles and $\frac{m+1}{2}$ number of $\{a^2\}$ tiles to create a graph of order $(\frac{m+1}{2} + m + 2)$ which is less than the order of BK_m .
- If $t_3 = \{\hat{a}^m, \hat{b}\}$, then $t_4 = \{\hat{b}^m, b\}$ and t_4 can connect with $(m - 1)$ number of $\{\hat{a}, b\}$ tiles and $\frac{m-1}{2}$ number of $\{a^2\}$ tiles to create a graph of order $\frac{m-1}{2} + m$ which is less than the order of BK_m .

Case 3b: Suppose $t_3 = \{\hat{a}^k, \hat{b}^j, *\}$ and $t_4 = \{\hat{a}^j \hat{b}^k, *\}$ where $k + j = m$. We now goes through the cases for $*$ and show that we will always be able to construct a smaller graph. We will have analogous proofs for when $*$ is b or \hat{b} so we will only consider when $*$ is a or \hat{a} . If $*$ is \hat{a} , then $t_3 = \{\hat{a}^{k+1}, \hat{b}^j\}$ and $t_4 = \{a, \hat{a}^j, \hat{b}^k\}$. If $*$ is a then $t_3 = \{a, \hat{a}^k, \hat{b}^j\}$ and $t_4 = \{\hat{a}^{j+1}, \hat{b}^k\}$. Notice that we will always have a case in which we have a tile of the form $\{\hat{a}^l, \hat{b}^p\}$ where $l + p = k + j + 1$. Thus, without loss of generality, suppose that $t_3 = \{\hat{a}^{k+1}, \hat{b}^j\}$, then $t_4 = \{a, \hat{a}^j, \hat{b}^k\}$. We can construct a smaller graph in the follow way. For every pair of \hat{a} half-edges of t_3 , use a t_1 . This will require $\lfloor \frac{k+1}{2} \rfloor$ number of t_1 's.

- If $k + 1$ is even, then j is even since $k + j + 1 = m + 1$ is even. Then we can use one t_3 tile, j number of t_2 tiles, and $\frac{k+1+j}{2}$ number of t_1 tiles to create a smaller graph of order $j + \frac{k+1+j}{2} + 1$ which is less than the target graph of $2(m + 1)$ or $2(j + k + 1)$.
- If $k + 1$ is odd, then j is odd since $k + j + 1 = m + 1$ is even. Then we can use one t_3 tile, $j - 1$ number of t_2 tiles, and $\frac{k+j-1}{2}$ number of t_1 tiles along with an additional t_1 and t_2 (used to close the remaining \hat{a} half edge and \hat{b} half edge). This creates a smaller graph of order $\frac{k+j-1}{2} + j + 2$ which is less than the target graph of $2(m + 1)$ or $2(j + k + 1)$.

Thus we can create a graph of order less than the target graph. Note if we added the tile $\{a, b\}$ to our pot, it would only be able to replace some of the $\{a^2\}$ tiles in the construction of our book graphs because our degree-2 vertices are adjacent and must be connected by a and \hat{a} (the latter which only comes from $\{\hat{a}, b\}$). As such our pot would still have $t_1 = \{a^2\}$, $t_2 = \{\hat{a}, b\}$ and at least one tile of the form $\{\hat{a}^l, \hat{b}^p\}$ where $l + p = k + j + 1$ and we would still be able to construct a smaller graph in the method described above. \square

Corollary 4.6 For BK_m where m is odd, $B_2(BK_m) = 3$ and $T_2(BK_m) = 4$.

Proof. By Theorem 4.5, $B_2(BK_m) \geq 3$. The pot $P = \{t_1 = \{a^m, c\}, t_2 = \{a^m, \hat{c}\}, t_3 = \{\hat{a}, b\}, t_4 = \{\hat{a}, \hat{b}\}\}$ will successfully realize any book graph under Scenario 2. Thus $B_2(BK_m) = 3$ and by Theorem 3.4, $T_2(BK_m) = 4$. The general construction matrix for BK_m is

$$M(P) = \left(\begin{array}{cccc|c} m & m & -1 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ \hline 1 & 1 & 1 & 1 & 1 \end{array} \right). \quad (2)$$

with solution set $\{(\frac{1}{2(m+1)}, \frac{1}{2(m+1)}, \frac{m}{2(m+1)}, \frac{m}{2(m+1)})\}$. \square



4.2 Stacked Book Graphs

Having thus addressed book graphs under Scenarios 1 and 2, we move on to stacked book graphs, formally defined in Definition 4.7. As the name of the family implies, the stacked book graphs are an extension of book graphs and appear as if several book graphs are stacked on top of each other. As such we will refer to many of the same terms we used with book graphs with stacked book graphs.

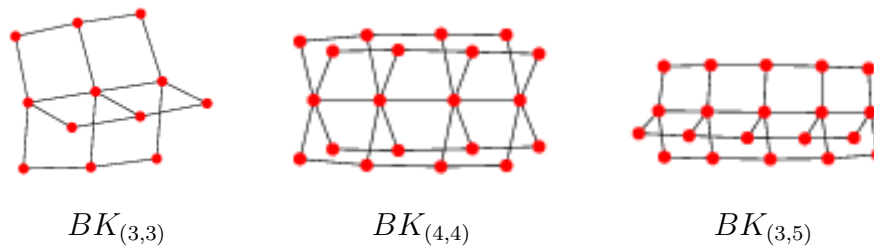


Figure 8: Stacked book family (pictures from [20])

Definition 4.7 *The stacked book graph of order (m, n) , $BK_{m,n}$, is defined as the graph Cartesian product $S_{m+1} \square P_n$ for $m \geq 0$ and $n \geq 1$, where S_{m+1} is a star graph and P_n is the path graph on n nodes. It is therefore the graph corresponding to the edges of $n - 1$ copies of an m -page “book” stacked one on top of another and is a generalization of the book graph.*

Something of note with the Stacked Book Family is that when there are more than two books with at least two pages, no matter how many books are stacked on top of each other, nor how many pages are added to the book, the distinct degrees in the sequence will always be $\{2, 3, m + 1, m + 2\}$. Note that for stacked book graphs of order $(0, n)$ are path graphs, $(1, n)$ are ladder graphs, $(2, n)$ are grid graphs, $(m, 1)$ are star graphs, and $(m, 2)$ are book graphs. Thus the smallest stacked book graph we will consider will have at least two books of three pages ($n \geq 3$ and $m \geq 3$).

Theorem 4.8 *Given stacked book graph $BK_{m,n}$ with $m, n \geq 3$, $T_1(BK_{m,n}) \leq 5$.*

Proof. Suppose G is a stacked book graph, $BK_{m,n}$ such that $m, n \geq 3$. One possible pot for G is $P = \{t_1 = \{a^2\}, t_2 = \{a, \hat{a}^2\}, t_3 = \{\hat{a}^{m+1}\}, t_4 = \{a^{m+1}, \hat{a}\}, t_5 = \{a^2, \hat{a}^m\}\}$, and thus $T_1(G) \leq 5$. \square

Theorem 4.9 *Given stacked book graph $BK_{m,n}$ such that $n = 3$ and $m \geq 3$, $T_1(BK_{m,n}) = 4$.*

Proof. Suppose G is a stacked book graph, $BK_{m,n}$, such that $n = 3$ and $m \geq 3$. Note that $av(G) = |\{2, 3, m + 1, m + 2\}| = 4$. The pot $P = \{t_1 = \{\hat{a}^2\}, t_2 = \{a^3\}, t_3 = \{\hat{a}^{m+2}\}, t_4 = \{a^{m+1}\}\}$ realizes G . Thus for $m \geq 3$, $T_1(G) = 4 = av(G)$ and thus P is an optimal pot which can construct $BK_{m,n}$ with $n = 3$ and $m \geq 3$ under Scenario 1 by Theorem 3.1. \square



Theorem 4.10 Given stacked book graph $BK_{m,n}$ such that $n \geq 4$ and m is an odd number greater than or equal to 3, $T_1(BK_{m,n}) = 4$.

Proof. Suppose G is a stacked book graph, $BK_{m,n}$, such that $n \geq 4$ and m is an odd number greater than or equal to 3. $av(G) = |\{2, 3, m + 1, m + 2\}| = 4$. The pot $P = \{t_1 = \{a, \hat{a}\}, t_2 = \{a^2, \hat{a}\}, t_3 = \{a, \hat{a}^{m+1}\}, t_4 = \{a^{\frac{m+1}{2}}, \hat{a}^{\frac{m+1}{2}}\}\}$ realizes G . Thus $T_1(G) = 4$ and P is an optimal pot which can construct $BK_{m,n}$ with odd m and $n \geq 4$ under Scenario 1 by Theorem 3.1. \square

Theorem 4.11 Given stacked book graph $BK_{m,n}$ such that n is odd and greater than or equal to 5, m is an even number greater than 3, and $(m - n + 1)$ is divisible by $(n - 2)$, $T_1(BK_{m,n}) = 4$.

Proof. Suppose G is a stacked book graph, $BK_{m,n}$, such that n is odd and greater than or equal to 5 and m is an even number greater than 3 such that $(m - n + 1)$ is divisible by $(n - 2)$. $av(G) = |\{2, 3, m + 1, m + 2\}| = 4$. The pot $P = \{t_1 = \{\hat{a}^2\}, t_2 = \{a^2, \hat{a}\}, t_3 = \{a^{m+1}\}, t_4 = \{a^{\frac{m+1-n}{n-2}+2}, \hat{a}^{m-\frac{m+1-n}{n-2}}\}\}$ realizes G . Thus $T_1(G) = 4$ and P is an optimal pot which can construct $BK_{m,n}$ such that n is odd and greater than or equal to 5, m is an even number greater than 3, and $(m - n + 1)$ is divisible by $(n - 2)$ under Scenario 1 by Theorem 3.1. \square

Figure 9 provides an example tiling for $BK_{4,5}$ and demonstrates the pattern between the a 's and \hat{a} 's. Note in Figure 9, the arrows point from a to \hat{a} , and for simplicity, the \hat{a} 's are represented by the heads of the arrows. It is an open problem whether or not other divisibility relationships yield similar results, but we are able to find an upper bound for graphs $BK_{m,n}$ such that n is odd and greater than or equal to 5, m is an even number greater than 3, and $(m - n + 1)$ is not divisible by $(n - 2)$ as stated below.

4.3 Crossed-prism Results

We now provide our results for the crossed-prism graph family.

Definition 4.12 An n -crossed-prism graph for a positive even n , CP_n , is a graph obtained by taking two disjoint cycle graphs C_n and adding edges (v_k, v_{k+1}) and (v_{k+1}, v_k) for $k = 1, 3, 5, \dots, (n - 1)$. This definition assumes that the vertices v_n and v_{n+1} are on the same cross for both the inner and outer cycles for all $v_n \in V$. See Figure 10.

Theorem 4.13 Given a crossed-prism graph, CP_n , $T_1(CP_n) = 2$ and $B_1(CP_n) = 1$ for all n .

Proof. Suppose we have a crossed-prism graph CP_n . Since CP_n is 3 regular by Theorems 3.3 and 3.2, $T_1(CP_n) = 2$ and $B_1(CP_n) = 1$. A pot which satisfies the requirements of Scenario 1 for CP_n is $P = \{t_1 = \{a, \hat{a}^2\}, t_2 = \{a^2, \hat{a}\}\}$. \square

Lemma 4.14 Given a crossed-prism graph, CP_n , $B_2(CP_n) \geq 2$ for all n .



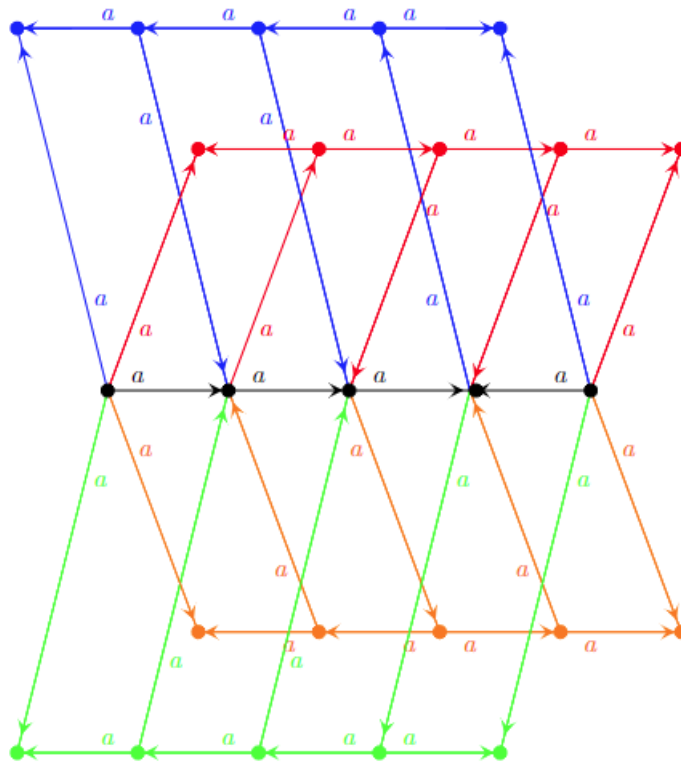


Figure 9: Example Tiling for $BK_{4,5}$ in Scenario 1. The arrows point from a to \hat{a} and for simplicity, the \hat{a} 's are represented by the heads of the arrows.

Proof. Assume $B_2(CP_n) = 1$. We cannot use the pot $P = \{t_1 = \{a, \hat{a}^2\}, t_2 = \{a^2 \hat{a}\}\}$ from Theorem 4.13 as it creates a graph of order two. Without loss of generality, the other option for a potential pot would be $P = \{t_1 = \{a^2, \hat{a}\}, t_2 = \{\hat{a}^3\}\}$ which creates a K_4 , the complete graph of four vertices (shown in Figure 12). Thus $B_2(CP_n) \geq 2$. \square

Corollary 4.15 For crossed-prism graph, CP_4 , $B_2(CP_4) = 2$ and $T_2(CP_4) = 3$.

Proof. From the previous Lemma, we know $B_2(CP_4) \geq 2$. A pot that realizes CP_4 in Scenario 2 is $P = \{t_1 = \{a^2, \hat{a}\}, t_2 = \{\hat{a}^2, b\}, t_3 = \{\hat{a}, \hat{b}^2\}\}$ (or $P = \{t_1 = \{a, \hat{a}^2\}, t_2 = \{a^2, b\}, t_3 = \{a, \hat{b}^2\}\}$) as shown in Figure 13) with solution set $S(P) = \{(\frac{5}{8}, \frac{1}{4}, \frac{1}{8})\}$. By Theorem 3.4, since $T_2(G) = B_2(G) + 1$, the number of tile types is minimal. \square

Corollary 4.16 Given a crossed-prism graph, CP_n , $B_2(CP_n) \leq \frac{n}{2}$ and $T_2(CP_n) \leq \frac{n}{2} + 1$, for all n . Note $2n$ is equal to the order of CP_n .

Proof.

A general pot for CP_n in Scenario 2 is given by $P = \{t_1 = \{a_1^2, \hat{a}_1\}, t_2 = \{\hat{a}_1^2, a_2\}, t_3 = \{a_1, \hat{a}_2, a_3\}, t_4 = \{\hat{a}_1, \hat{a}_3, a_4\}, t_5 = \{\hat{a}_1, \hat{a}_4, a_5\}, \dots, t_{m+1} = \{\hat{a}_1, \hat{a}_m, a_{m+1}\}, t_{m+2} = \{\hat{a}_1, \hat{a}_{m+1}, a_{m+2}\}, \dots, t_{\frac{n}{2}} = \{\hat{a}_1, \hat{a}_{\frac{n}{2}-1}, a_{\frac{n}{2}}\}, t_{\frac{n}{2}+1} = \{\hat{a}_1, \hat{a}_{\frac{n}{2}}\}\}$ with spectrum



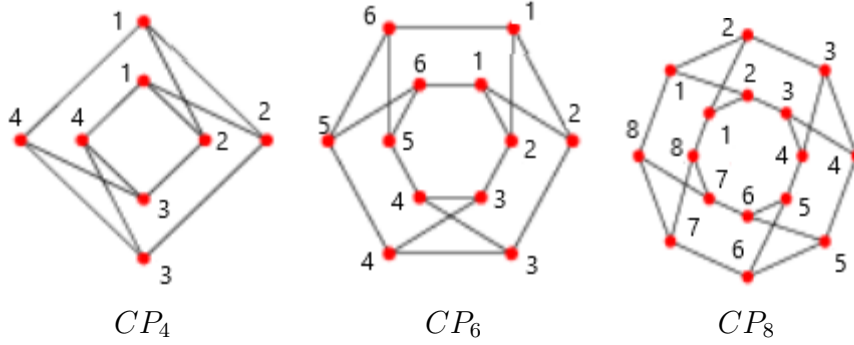


Figure 10: Crossed-Prism Family (Pictures from [17])

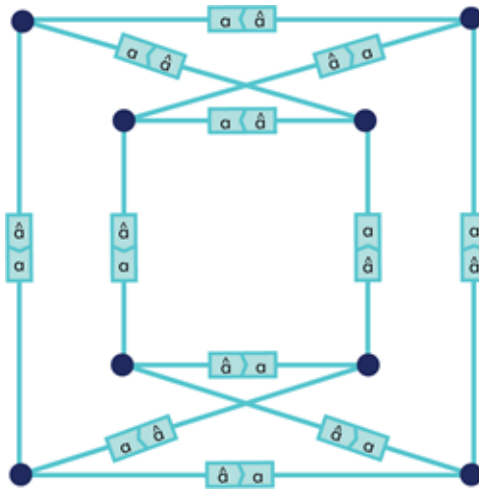


Figure 11: Size 8 crossed-prism graph

$S(P) = \{(\frac{n+1}{2n}, \frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n}, \frac{1}{2n})\}$ and thus no graphs of order less than $2n$ are made. Thus $B_2(CP_n) \leq \frac{n}{2}$ and $T_2(CP_n) \leq \frac{n}{2} + 1$. \square

Example: The pot for a CP_6 in Scenario 2 is given by $P = \{t_1 = \{a^2, \hat{a}\}, t_2 = \{\hat{a}^2, b\}, t_3 = \{\hat{a}, \hat{b}, c\}, t_4 = \{\hat{a}, \hat{c}^2\}\}$ with spectrum $S(P) = \{(\frac{7}{12}, \frac{1}{6}, \frac{1}{6}, \frac{1}{12})\}$ as shown in Figure 14. The pot for CP_8 is $P = \{t_1 = \{a^2, \hat{a}\}, t_2 = \{\hat{a}^2, b\}, t_3 = \{\hat{a}, \hat{b}, c\}, t_4 = \{\hat{a}, \hat{c}, d\}, t_5 = \{\hat{a}, \hat{d}^2\}\}$ with spectrum $S(P) = \{(\frac{9}{16}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{16})\}$. The pot for CP_{10} is $P = \{t_1 = \{a^2, \hat{a}\}, t_2 = \{\hat{a}^2, b\}, t_3 = \{\hat{a}, \hat{b}, c\}, t_4 = \{\hat{a}, \hat{c}, d\}, t_5 = \{\hat{a}, \hat{d}, e\}, t_6 = \{\hat{a}, \hat{e}^2\}\}$ with solution set $S(P) = \{(\frac{11}{20}, \frac{1}{10}, \frac{1}{10}, \frac{1}{10}, \frac{1}{10}, \frac{1}{20})\}$. Note that the pot stays the same as the crossed-prism graph grows except for the last two tiles.

4.4 Crown Graphs

The final family of bipartite graphs examined in this research is the family of crown graphs. Crown graphs are similar to complete bipartite graphs which have two columns with the same number of vertices that connect to every vertex in their opposite column. However in a crown graph, a given vertex does not connect its corresponding vertex in



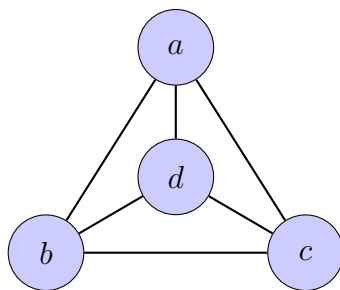


Figure 12: K_4 .

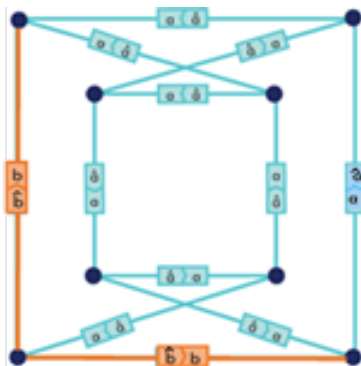


Figure 13: Crossed-prism graph

the opposite column. Figure 15 gives an example of crown graphs ranging from order 6 to order 12.

Definition 4.17 An n -crown graph, Cr_n , is $Cr_n = \overline{K_n \square K_2}$, the complement of the Cartesian direct product of the complete graph of order n and the complete graph of order 2.

Lemma 4.18 For crown graph Cr_n , $B_1(Cr_n) = 1$ and $T_1(Cr_n) = 1$ for odd n and $T_1(Cr_n) = 2$ for even n .

Proof. It follows from Theorem 3.2 that $B_1(Cr_n) = 1$. Note that crown graphs are regular and all vertices are of degree $(n - 1)$. When n is odd, $(n - 1)$ will be even. Therefore, by Theorem 3.3 $T_1(Cr_n) = 1$. When n is even, $(n - 1)$ will be odd. Therefore by Theorem 3.3, $T_1(Cr_n) = 2$. \square

Lemma 4.19 For crown graph Cr_n , $T_2(Cr_n) \geq 2$ for all n .

Proof. Let Cr_n be a crown graph. If n is even, then $2 = T_1(Cr_n) \leq T_2(Cr_n)$. Suppose n is odd, then the only possible way to tile Cr_n with one tile, would be to use $t_1 = \{a^{\frac{n-1}{2}}, \hat{a}^{\frac{n-1}{2}}\}$ which creates a graph of order one. Thus $T_2(Cr_n) \geq 2$. \square



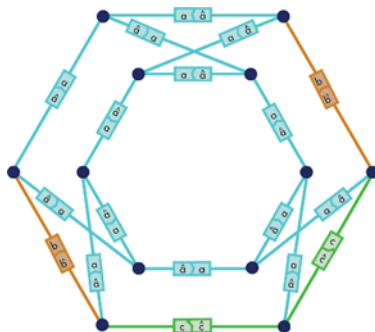


Figure 14: CP_6 Crossed-prism graph

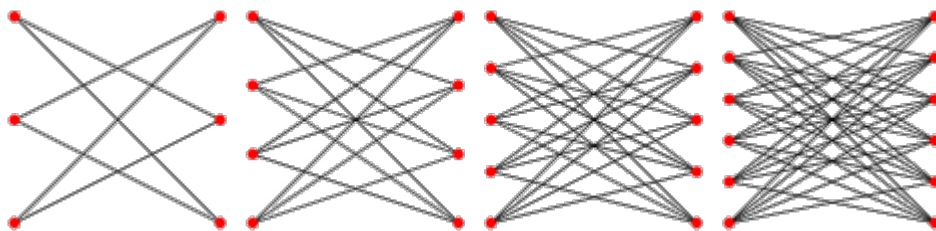


Figure 15: Crown graph family (picture from [18])

Theorem 4.20 For crown graph Cr_n , $B_2(Cr_n) > 1$ for all n .

Proof. Assume $B_2(Cr_n) = 1$.

Since crown graphs are $(n - 1)$ -regular, the only possible values for the net sum of a 's and \hat{a} 's on each vertex, d_{n-1} , are $d_{n-1} \in \{\pm 1, \pm 2, \dots, \pm(n - 1)\}$. Note when n is odd, $d_{n-1} \neq 0$ since this will create a graph of order 1 (with each a half edge connecting to one of the \hat{a} half edges). When n is even, then $n - 1$ is odd and 0 is not a possible value for d_{n-1} . The construction matrix for $p \leq 2n$ is given by

$$M(P) = \left(\begin{array}{cccc|c} z_{1,1} & z_{1,2} & \dots & z_{1,p} & 0 \\ 1 & 1 & 1 & 1 & 1 \end{array} \right). \quad (3)$$

Since the number of a and \hat{a} must be equivalent, without loss of generality, assume $z_{1,1} > 0$ and $z_{1,2} < 0$. Thus $M(P)$ is row equivalent to

$$\left(\begin{array}{cccc|c} 1 & 0 & -\frac{-z_{1,3}+z_{1,2}}{z_{1,1}-z_{1,2}} & \dots & -\frac{-z_{1,p}+z_{1,2}}{z_{1,1}-z_{1,2}} & -\frac{z_{1,2}}{z_{1,1}-z_{1,2}} \\ 0 & 1 & \frac{z_{1,1}-z_{1,3}}{z_{1,1}-z_{1,2}} & \dots & \frac{z_{1,1}-z_{1,p}}{z_{1,1}-z_{1,2}} & \frac{z_{1,1}}{z_{1,1}-z_{1,2}} \end{array} \right). \quad (4)$$

which has a solution of the form $(\frac{-z_{1,2}}{z_{1,1}-z_{1,2}}, \frac{z_{1,1}}{z_{1,1}-z_{1,2}}, 0, \dots, 0)$. Let $m = z_{1,1} - z_{1,2}$. Since $z_{1,1} > 0$ and $z_{1,2} < 0$, and due the constraints on d_{n-1} , we have $m \leq 2(n - 1) < 2n$. Thus a graph of order less than $2n$ can be made. \square

Lemma 4.21 $B_2(Cr_3) = 3$.



Proof. By Lemma 4.20, $B_2(Cr_3) \geq 2$. Assume $B_2(Cr_3) = 2$. First we show that we cannot use tiles with two different bond-edge types for each vertex. Without loss of generality, Assume we have $t_1 = \{a, b\}$ and $t_2 = \{\hat{a}, \hat{b}\}$. This requires $t_3 = \{\hat{a}, \hat{b}\}$ or $t_3 = \{a, \hat{b}\}$. Both options produce graphs of order 2.

We now continue with the assumption that we use at least one tile with a repeated letter. Without loss of generality, let $t_1 = \{a^2\}$, $t_2 = \{\hat{a}, b\}$. This means we need a tile, say t_3 , which uses \hat{b} . Since t_3 cannot use b as this will create a graph of order 1, we have the following other choices for the second edge of t_3 : a, \hat{a} , or \hat{b} .

Case 1: Suppose $t_1 = \{a^2\}$, $t_2 = \{\hat{a}, b\}$, and $t_3 = \{a, \hat{b}\}$. This creates a graph of order 2 using t_2 and t_3 .

Case 2: Suppose $t_1 = \{a^2\}$, $t_2 = \{\hat{a}, b\}$, and $t_3 = \{\hat{a}, \hat{b}\}$. This creates a graph of order 3 using each tile once.

Case 3: Suppose $t_1 = \{a^2\}$, $t_2 = \{\hat{a}, b\}$, and $t_3 = \{\hat{b}^2\}$. This creates a graph of order 4 using one t_1 , one t_3 , and two t_2 's.

Therefore $B_2(Cr_3) \geq 3$. The pot $P = \{t_1 = \{a^2\}, t_2 = \{b^2\}, t_3 = \{\hat{a}, c\}, t_4 = \{\hat{b}, \hat{c}\}\}$ realizes Cr_3 . The Spectrum for the construction matrix associated with this pot is $S(P) = \{(\frac{1}{6}, \frac{1}{6}, \frac{2}{6}, \frac{2}{6})\}$ and thus no smaller graphs can be made and so $T_2(Cr_3) = 4$. \square

Lemma 4.22 $B_2(Cr_4) = 2, T_2(Cr_4) = 3$.

Note that Cr_4 isomorphic to the cube and CP_4 so our pot in Corollary 4.15 works for this. The following pot realizes the cube/ Cr_4 : $P = \{t_1 = \{a, b^2\}, t_2 = \{a^2, \hat{b}\}, t_3 = \{a, \hat{a}^2\}\}$. See Figure 16.

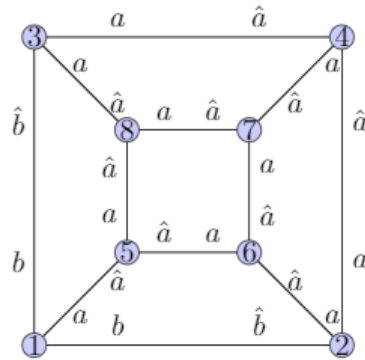


Figure 16: Cube/ CP_4 / Cr_4 in Scenario 2

We were able to find upper bounds for the optimal number of bond-edge types and tiles for crown graphs.

Lemma 4.23 For crown graph Cr_n , $B_2(Cr_n) \leq 3$ and $3 \leq T_2(Cr_n) \leq 4$ for all n .

Proof. By Theorem 3.4, we know $T_2(Cr_n) \geq 3$. The following pot realizes Cr_n for any



n : $P = \{\{a^{n-1}\}, \{b^{n-1}\}, \{\hat{a}, c^{n-2}\}, \{\hat{b}, \hat{c}^{n-2}\}\}$. The general construction matrix for P is

$$M(P) = \left(\begin{array}{cccc|c} n-1 & 0 & -1 & 0 & 0 \\ 0 & n-1 & 0 & -1 & 0 \\ 0 & 0 & n-2 & 2-n & 0 \\ \hline 1 & 1 & 1 & 1 & 1 \end{array} \right)$$

with solution set $S = \{\frac{1}{2n}, \frac{1}{2n}, \frac{n-1}{2n}, \frac{n-1}{2n}\}$ and thus no smaller graphs can be made. Therefore $B_2(Cr_n) \leq 3$ and $T_2(Cr_n) \leq 4$. \square

4.5 Other Graphs

We also explored various other graphs which are not bipartite graphs.

4.5.1 Helm Graphs

A *helm graph*, H_n , is obtained using the base of an n -order wheel graph and adding a pendant edge to each outer vertex [19]. Helm graphs have an order of $2n + 1$ and size $3n$. Examples of helm graphs are shown in Figure 17.

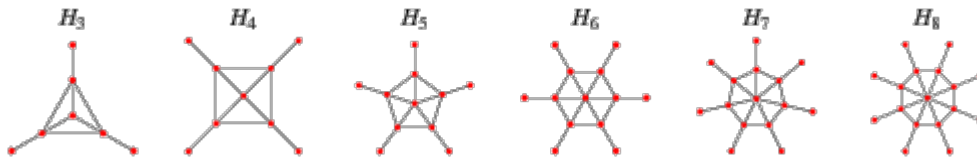


Figure 17: Helm graph family (picture from [19])

Theorem 4.24 For helm graph, H_n , $T_1(H_n) = 3$ for all n .

Proof. We have two cases since we have a different number of degree types when $n = 4$ due to the structure of helm graphs.

Case 1: For H_n where $n \neq 4$, since $av(H_n) = 3$, it follows from Theorem 3.1 that $T_1(H_n) \geq 3$. The following pot realizes H_n graphs in Scenario 1: $P = \{t_1 = \{a\}, t_2 = \{a^2, \hat{a}^2\}, t_3 = \{\hat{a}^n\}\}$. An example of the tiling can be found in Figure 18.

Case 2: For H_4 $av(H_4) = 2$, $ev(H_4) = 1$, and $ov(H_4) = 1$. Thus it follows from Theorem 3.1 that $2 \leq T_1(H_4) \leq 3$. Assume $T_1(H_4) = 2$ and there exists a pot P' which uses only 2 tiles to realize H_4 in Scenario 1. Without loss of generality, assume $t_1 = \{a\}$. We then have three cases for the choice of t_2 . Recall, d_1 and d_4 denote the net total of a 's and \hat{a} 's for t_1 and t_2 respectively. Since the net number of a 's and \hat{a} must be equivalent we must satisfy the following equation:

$$4d_1 + 5d_4 = 0 \tag{5}$$

Since $d_1 = 1$, this equation reduces to $4 + 5d_4 = 0$ and thus $d_4 = \frac{-4}{5}$ and thus has no integer solutions and thus $T_1(H_4) > 2$. The following pot P realizes H_4 in Scenario 1: $P = \{t_1 = \{a\}, t_2 = \{a^2, \hat{a}^2\}, t_3 = \{\hat{a}^4\}\}$. \square



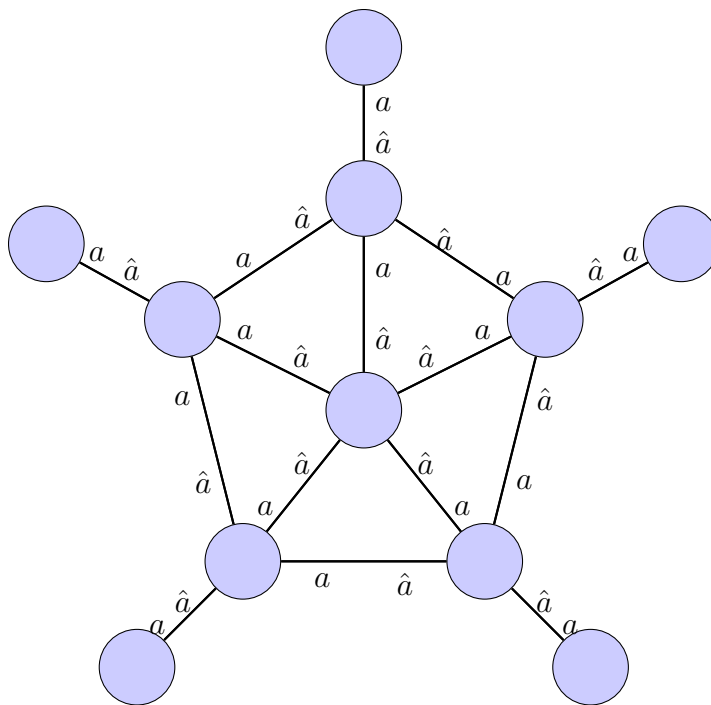


Figure 18: Tiling of the H_5 in Scenario 1

Theorem 4.25 For helm graph H_n , $B_2(H_n) \geq 2$ for all n .

Proof. Assume $B_2(H_n) = 1$. We will consider the possible options for our degree 4 tile which we will denote as t_1 .

Case 1: If $t_1 = \{a^2, \hat{a}^2\}$, this will create a graph of order 1.

Case 2: If $t_1 = \{a^4\}$, since our degree 4 vertex is adjacent to the degree 1 vertex, this means our degree 1 vertex must be $\{\hat{a}\}$. And thus we can create a graph of order 5 which is less than the order of any helm graph.

Case 3: Without loss of generality, suppose $t_1 = \{a, \hat{a}^3\}$ (note if $t_1 = \{a^3, \hat{a}\}$, the argument is analogous). To satisfy the requirement that the total number of hatted and unhatted cohesive end types must be equal and because our degree 4 vertices are adjacent (and thus requiring them to connect their half edges with a 's and \hat{a} 's), our other tiles are forced to be $t_2 = \{a\}$ and $t_3 = \{a^n\}$. Thus we have the pot $P' = \{t_1 = \{a, \hat{a}^3\}, t_2 = \{a\}, t_3 = \{a^n\}\}$. Unfortunately, this pot realizes graphs of smaller order since we can use one t_1 and two t_2 to make a graph of order 3.

Therefore $B_2(H_n) \geq 2$.

□

Corollary 4.26 For helm graph H_n , $B_2(H_n) = 2$ and $T_2(H_n) = 3$ for all n .

Proof. By Theorem 4.25, $B_2(H_n) \geq 2$. The following pot realizes H_n graphs in Scenario



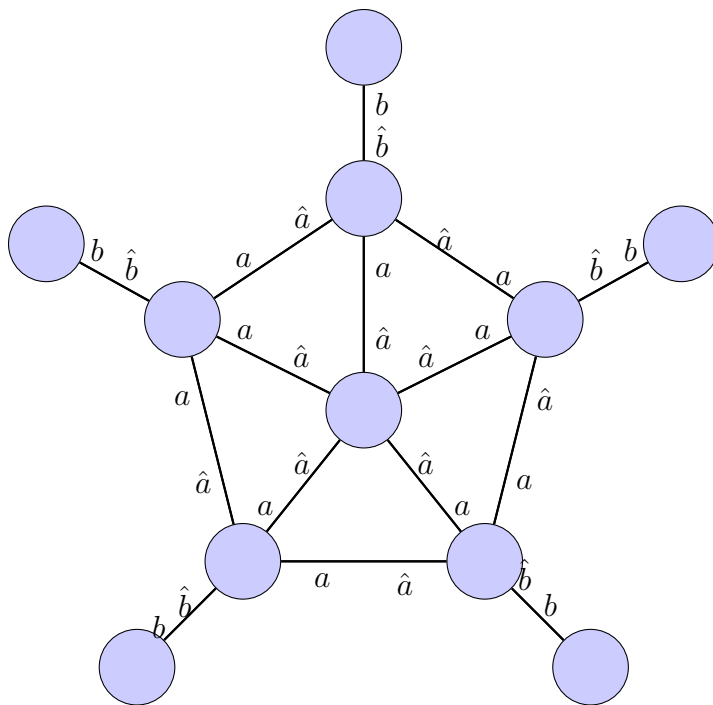


Figure 19: Tiling of the H_5 in Scenario 2

2: $P = \{t_1 = \{\hat{a}^n\}, t_2 = \{a^2, \hat{a}, \hat{b}\}, t_3 = \{b\}\}$. The construction matrix for this pot is

$$M(P) = \left(\begin{array}{ccc|c} -n & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ \hline 1 & 1 & 1 & 1 \end{array} \right)$$

which gives the solution set $S = \{\frac{1}{2n+1}, \frac{n}{2n+1}, \frac{n}{2n+1}\}$. Therefore no smaller graphs can be realized and $B_2(H_n) = 2$. By Theorem 3.4, since $B_2(G) + 1 \leq T_2(G)$, $T_2(H_n) = 3$. \square

4.5.2 Moth, Cricket, and Kite Graphs

In this section, we share our results for the moth graph, M , cricket graph, C , and kite graph, K . Several of these graphs have interesting properties. For example, the moth graph's Scenario includes a free variable and the kite graph is an example in which we found two different pots to minimize bond-edge types and tile types.

The *moth graph* is the 6-vertex graph as shown in Figure 20.

Theorem 4.27 For the moth graph, M , $T_1(M) = 4$.

Proof. By Theorem 3.1, $4 \leq T_1(M) \leq 7$. The moth graph can be realized by the pot $P = \{t_1 = \{a^2\}, t_2 = \{a, \hat{a}^2\}, t_3 = \{a\}, t_4 = \{\hat{a}^5\}\}$ and thus $T_1(M) = 4$. \square



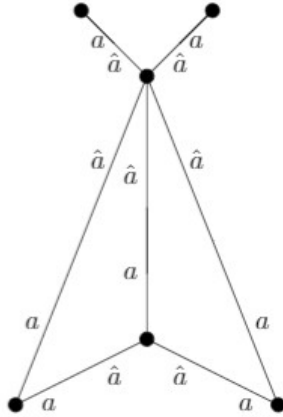


Figure 20: Labeled moth graph in Scenario 1

Theorem 4.28 For the moth graph, M , $T_2(M) = 4$.

Proof. $T_1(G) \leq T_2(G)$ for all graphs G [5]. Thus by the previous theorem, $T_2(M) \geq T_1(M) = 4$. The moth graph can be realized with four tile types by the pot $P = \{t_1 = \{a\}, t_2 = \{a, b\}, t_3 = \{\hat{b}^3\}, t_4 = \{\hat{a}^4, b\}\}$. We can use the construction matrix to show that this pot does not realize smaller graphs.

$$M(P) = \left(\begin{array}{cccc|c} 1 & 1 & 0 & -4 & 0 \\ 0 & 1 & -3 & 1 & 0 \\ \hline 1 & 1 & 1 & 1 & 1 \end{array} \right).$$

Note that this construction matrix will have a free variable. Using a Maple program from [2] and given the restrictions on the free variable within this application of the construction matrix, we found the solution set of this pot to be $S(P) = \{\frac{1}{r} \langle -3r + 20t, 3r - 16t, r - 5t, t \rangle \mid r \in \mathbb{Z}^+, t \in (\mathbb{Z} \cap [\frac{3r}{20}, \frac{3r}{16}])\}$. Using the program from [2], it can be shown that the order of the smallest graph which can be realized from this pot of tiles is 6 with $r = 6$ and $t = 1$ and the solution set gives $\{(\frac{1}{3}, \frac{1}{3}, \frac{1}{6}, \frac{1}{6})\}$, thus no graph with order less than 6 can be made.

□

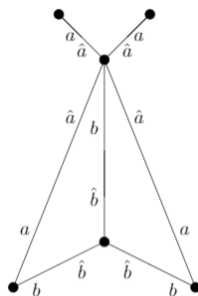


Figure 21: Labeled moth graph in Scenario 2



Lemma 4.29 Suppose G is a graph of order greater than 3. If $B_2(G) = 1$ and G has both a degree-2 tile and a degree-1 tile, then both tiles must be, without loss of generality, $\{a^2\}$ and $\{a\}$.

Proof. Let G be a graph of order greater than three with at least one degree-2 vertex and one degree-1 vertex and suppose $B_2(G) = 1$. Without loss of generality, the degree-2 tile must be of the form $\{a^2\}$ since $\{a, \hat{a}\}$ creates a smaller graph. This forces our tiles with one edge to also be tiled $\{a\}$, otherwise, these two tile types realize a smaller graph of order three. \square

Theorem 4.30 For the moth graph, M , $B_2(M) = 2$.

Proof. Assume $B_2(M) = 1$. By Lemma 4.29, our degree-2 vertex must be tiled $\{a^2\}$ and our degree-1 vertex must be tiled $\{a\}$. The tile with five edges now has four \hat{a} 's because of its connections to the other two tile types. The missing edge cannot be labeled with \hat{a} as this can create a graph of order four using two $\{a^2\}$, one $\{a\}$, and one $\{\hat{a}^5\}$. We can also not use a for this edge since it can create a graph of order three using one $\{a^2\}$ tiles, one $\{a\}$ tile, and one $\{a, \hat{a}^4\}$ tile. Thus $B_2(M) \geq 2$. The pot in Theorem 4.28 gives a pot that realizes the moth graph using only 2 bond-edge types. \square

4.5.3 Cricket Graph

The *cricket graph*, C , is a 5-vertex graph shown in Figure 22.

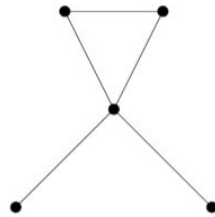


Figure 22: Cricket graph

Theorem 4.31 For the cricket graph, C , $T_1(C) = 3$.

Proof. By Theorem 3.1, $3 \leq T_1(C) \leq 4$. The cricket can be realized using three tile types with the pot $P = \{t_1 = \{a\}, t_2 = \{a, \hat{a}^3\}, t_3 = \{a, \hat{a}\}\}$. Note $B_1(C) = 1$ by Theorem 3.2. \square

Lemma 4.32 In Scenario 2, adjacent degree-2 vertices cannot be repeated and require at least two bond types.

Proof. Suppose we have two adjacent vertices both of degree two. Without loss of generality, let one vertex be tiled with $\{a^2\}$. Since the vertices are adjacent, the other vertex must have a complement \hat{a} . So we get two cases for the construction. Using the tile $\{\hat{a}, a\}$ creates a graph of order one, and using the tile $\{\hat{a}^2\}$ creates a graph of order two



when connecting with the original tile. Therefore we must introduce another bond-edge type and the adjacent degree-2 vertices must be distinct. \square

Theorem 4.33 For the cricket graph, C , $B_2(C) = 2$ and $T_2(C) = 4$.

Proof. Since we have two adjacent vertices of degree two, by Lemma 4.32, we must use more than one bond-edge type. Thus $B_2(C) \geq 2$.

$T_2(C) \geq T_1(C) = 3$ [5]. Suppose $T_2(C) = 3$. Lemma 4.32 states that we cannot repeat the adjacent degree-2 tiles. Therefore the number of tiles used is at least four. The cricket graph can be realized with four tile types with the pot $P = \{t_1 = \{a\}, t_2 = \{a, b\}, t_3 = \{a, \hat{b}\}, t_4 = \{\hat{a}^4\}\}$. The following is the construction matrix for this pot:

$$M(P) = \left(\begin{array}{cccc|c} 1 & 1 & 1 & -4 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ \hline 1 & 1 & 1 & 1 & 1 \end{array} \right).$$

Note that this construction matrix will have a free variable. Using a Maple program from [2], we found the solution set of this pot to be $S(P) = \{(0, \frac{2}{5}, \frac{2}{5}, \frac{1}{5}), (\frac{4}{5}, 0, 0, \frac{1}{5}), (\frac{2}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5})\}$ and thus no graphs of order less than 5 can be made. Therefore $T_2(C) = 4$ and $B_2(C) = 2$. \square

4.5.4 Kite Graph

The *kite graph* is a graph of order five which is shown in Figure 23.

Theorem 4.34 For the kite graph, K , $T_1(K) = 3$.

Proof. By Theorem 3.1, $3 \leq T_1(K) \leq 5$. The kite graph can be realized using three tile types with the pot $P = \{t_1 = \{a\}, t_2 = \{a^2\}, t_3 = \{a, \hat{a}^2\}\}$ as shown in Figure 23. \square

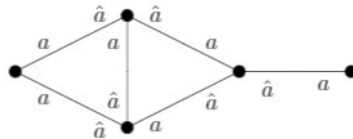


Figure 23: Kite graph in Scenario 1

Theorem 4.35 For the kite graph, K , $T_2(K) = 4$.

Proof. Assume $T_2(K) = 3$. Since $av(K) = 3$, in order to optimize our tile types, we must use the same tile for all degree-3 vertices. The kite graph has three adjacent vertices of degree-3 which means that the tile $\{a^3\}$ cannot be used in the pot if we are using the same repeated tile for the degree-3 vertices. Therefore we have four tile types that can be used in the pot to create a graph with three adjacent degree-3 vertices, $\{a^2, \hat{a}\}, \{a, \hat{a}^2\}, \{a, \hat{a}, b\}, \{a, \hat{a}, \hat{b}\}$. However, the degree-1 tile that connects to one of the degree-3 tiles will always create a smaller graph of order 2. Thus, $T_2(K) > 3$. The kite



graph can be realized in four tiles with the pot $P = \{t_1 = \{\hat{a}, b\}, t_2 = \{a, \hat{a}, \hat{b}\}, t_3 = \{a, b, \hat{c}\}, t_4 = \{c\}\}$. The following is the construction matrix for this pot:

$$M(P) = \left(\begin{array}{cccc|c} -1 & 0 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ \hline 1 & 1 & 1 & 1 & 1 \end{array} \right)$$

with solution set $S(P) = \{(\frac{1}{5}, \frac{2}{5}, \frac{1}{5}, \frac{1}{5})\}$ and thus no graphs of order less than 5 can be achieved. \square

Theorem 4.36 *For the kite graph, K , $B_2(K) = 2$.*

Proof. Suppose $B_2(K) = 1$. By Lemma 4.29, our single degree-2 vertex must be tiled $\{a^2\}$ and our single degree-1 vertex must be tiled $\{a\}$. This means all three of our degree-3 tiles must have at least one \hat{a} bond-edge label. Since these degree-3 vertices are adjacent, without loss of generality, we will label the edge across the center of the kite ($a\hat{a}$) so one degree-3 vertex has at least two \hat{a} 's. Notice that we cannot use the tile $\{a, \hat{a}^2\}$ because it creates a graph of order two with the degree-1 vertex. Thus our degree-3 vertex with at least two \hat{a} 's must be $\{\hat{a}^3\}$. This forces the degree-3 vertex connected to the degree-1 vertex to have at least one a and one \hat{a} . Since we can't have a second \hat{a} , this vertex must be tiled using $\{a^2\hat{a}\}$, but this forces the remaining degree-3 vertex to be $\{a, \hat{a}^2\}$ which cannot be as it creates a graph of order two. Thus $B_2(K) \geq 2$. The following pot realizes the kite graph: $P = \{t_1 = \{a^3\}, t_2 = \{\hat{a}^2\}, t_3 = \{a, \hat{a}, b\}, t_4 = \{\hat{a}, \hat{b}^2\}, t_5 = \{b\}\}$. The following is the construction matrix for this pot:

$$M(P) = \left(\begin{array}{ccccc|c} 3 & -2 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 \\ \hline 1 & 1 & 1 & 1 & 1 & 1 \end{array} \right).$$

Note this construction matrix has a free variable. Using the software from [2], the possible solutions of this pot are: $S(P) = \{(\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}), (\frac{1}{5}, \frac{1}{5}, 0, \frac{1}{5}, \frac{2}{5}), (\frac{1}{5}, \frac{1}{5}, \frac{2}{5}, \frac{1}{5}, 0), (\frac{3}{5}, \frac{2}{5}, 0, 0, 0)\}$. Thus no graphs of order less than 5 can be created and therefore $B_2(K) = 2$. \square

5 Conclusion and Open Questions

We have proved optimal tiling strategies in Scenario 1 for the following bipartite graph families: crossed-prism graphs, crown graphs, book graphs, and stacked book graphs. We found optimal pots for book graphs in Scenario 2 and found upper bounds for design strategies for crossed-prism graphs and crown graphs in Scenario 2. We also proved optimal construction strategies in Scenarios 1 and 2 for non-bipartite graphs kite, cricket, moth, and the helm graph families. Several of these graphs had interesting properties. The moth and kite graph provided examples in which the construction matrix had a free variable. Furthermore, the kite graph provides an example in which we found two different pots to minimize bond-edge types and tile types. Since this research focused on design strategies for Scenarios 1 and 2, Scenario 3 remains open for each of these graphs.



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