The KnotLink Game

H. Adams, A. Henrich, and S. Stoll

Abstract - Recently, several new games have been introduced that can be played on knot and link diagrams. One of the first such games, played on knot diagrams, is called the Knotting-Unknotting Game. In this game, one player aims to create an unknot while their opponent tries to produce a nontrivial knot. The Linking-Unlinking Game is similar, but is played on link diagrams. In this game, one player's goal is to produce an unlink while the other player aims to create any nontrivial link. In our paper, we introduce a hybrid of these two games, called the KnotLink game, that can be played on either a knot or a link diagram. Moves and players' goals are similar to those of the previous two games, with one key difference that allows the game board to be transformed from a knot to a link or vice versa during game play. We describe this new game, provide a sample game, and prove several results regarding winning strategies for infinite families of rational knots and links.

Keywords : knot; link; combinatorial game; unknotting

Mathematics Subject Classification (2020) **:** 57K10; 91A05

1 Introduction

While playing a game is usually considered to be a recreational activity, sometimes it can actually be viewed as mathematics research. That was the case for us. In particular, there are several interesting games that can be played on knot and link diagrams. While we have enjoyed playing some of these games that have been introduced by other teams of math researchers, doing so inspired us to invent a new game that is a fascinating hybrid of two games that have previously been studied. In this paper, we introduce our new game, called the KnotLink game, and study it from the perspective of combinatorial game theory, using facts from knot theory as we delve into winning strategies. Before we begin, though, let's acquaint ourselves with some essential background material from the world of mathematical knots and links.

1.1 Knot Theory Essentials

A **knot** is an embedding of a circle into 3-dimensional Euclidean space. Knots are often represented by **knot diagrams**, or 2-dimensional closed curves with a finite number of transverse self-intersections. Crossings in knot diagrams are decorated so that it is clear which strand should pass over and which should pass under at each intersection point. The special knot that can be represented by a diagram with no crossings is called the **unknot**.

All other knots are called **nontrivial knots**. Similarly, **links** are—possibly intertwining—collections of knots, where each knot in the collection is called a **component** of the link. A **link diagram** is a 2-dimensional representation of a link. If a link is simply a disjoint collection of unknots, we call it an **unlink**.

Knot theory concerns the study of knots and links. In particular, the central question of knot theory deals with classification: when are two knots or links the same and when are they different? We give an example of a nontrivial knot (the **trefoil**) and a nontrivial link (the **Hopf link**) in Figure 1.

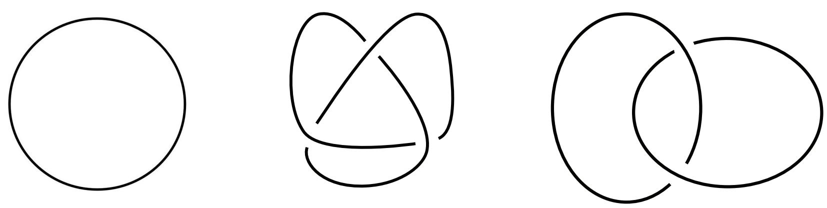

Figure 1: A diagram of the unknot (left), the trefoil knot (center) and a Hopf link (right)

One tool knot theorists use to help them determine if two knots or links are the same is Reidemeister's theorem. In the 1920's, Kurt Reidemeister [8] and, independently, Alexander and Briggs [1] proved that two knot or link diagrams represent the same knot or link *if and only if* the diagrams can be related by a sequence of Reidemeister moves. The three Reidemeister moves are shown in Figure 2. As we'll see later, these moves are very useful when we need to identify whether or not a given knot or link diagram represents an unknot/unlink or a nontrivial knot/link.

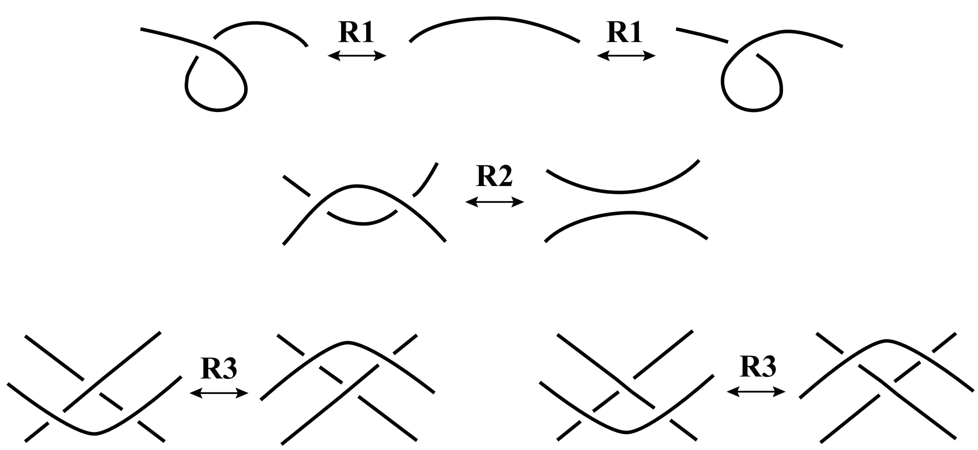

Figure 2: The Reidemeister moves: R1, R2, and R3.

1.2 Games Played on Knot and Link Diagrams

We have enjoyed playing several two-player combinatorial games inspired by knot theory [2, 3, 4, 5, 6]. The focus of this paper, however, is a new type of knot game, called the KnotLink Game. This game is a hybrid of two previously studied games, called the Knotting-Unknotting Game [6] and the Linking-Unlinking Game [3].

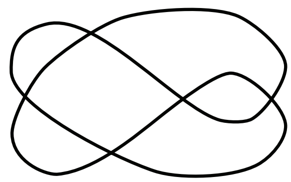

Figure 3: A knot shadow, or game board

In the Knotting-Unknotting Game [6], two players, the Knotter and the Unknotter, are given a game board, which is the **shadow** of a knot. A knot shadow is the projection of a knot onto a plane in such a way that over-under crossing information is undetermined. See, for example, Figure 3. In this game, players take turns **resolving** crossings. In other words, a move consists of choosing an unknown crossing and deciding which strand goes over and which goes under at that crossing. Once all of the crossings have been resolved, the resulting knot is either an unknot or is nontrivially knotted. The goal for the Knotter is to turn the game board into any nontrivial knot, while the goal of the Unknotter is to turn the game board into an unknot.

Note that, during game play, the game board may be neither a shadow nor a knot diagram. Such diagrams, where only some of the crossing information is known, are called **pseudodiagrams**. If a pseudodiagram will turn into an unknot regardless of how its unknown crossings are resolved, we call it an **unknotted pseudodiagram**. An example is shown in Figure 4.

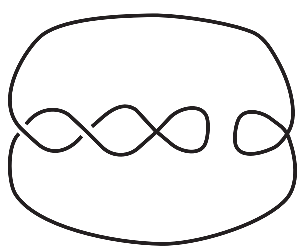

Figure 4: An unknotted pseudodiagram

The Knotting-Unknotting Game inspired the invention of another game, the Linking-Unlinking Game [3], where players are given a link shadow. Once again, players take turns resolving unknown crossings. In this game, one player, the Unlinker, wants to create the unlink while the other player, the Linker, aims to create any nontrivial link.

In this paper, we will take inspiration from both the Linking-Unlinking Game and the Knotting-Unknotting Game. We combine these two games to invent a game that can be played on either a knot shadow or a link shadow. In our game, called the KnotLink Game, the Simplifier plays with the goal of creating either an unknot or an unlink, and the Complicator's goal is to make any nontrivial knot or link. Most moves in the game consist of players taking turns resolving crossings, but each player has a special move they have the option to use once, at any point in the game. This special move is *smoothing* an unknown crossing. See Figure 5. A player may opt to smooth at a crossing either horizontally or vertically, as shown in the figure.

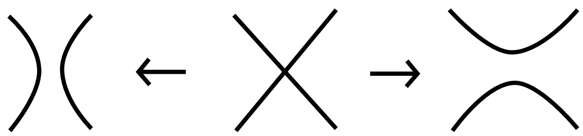

Figure 5: Smoothing an unknown crossing

Now that we have a new game to explore, our main aim is to determine which player has a **winning strategy** on certain knot and link shadows. In other words, given a starting game board and a choice of which player moves first, we'd like to know which player has a strategy that will guarantee them a win. For the purposes of this paper, we will consider game boards that are the shadows of standard minimal crossing diagrams of knots and links from two special families. One family is called the $(2, p)$**-torus knots and links**. The other is a particular family of rational knots and links, which we will call (p, q) **rational knots and links**. These families are best described via their standard diagrams. In Figure 6, we illustrate a diagram for $(2, p)$ torus knots and links, and in Figure 7, we illustrate a diagram for (p, q) rational knots and links. We see that the $(2, p)$ torus knots and links are formed by taking a **twist** containing p crossings and closing the twist up to form a knot or link. The (p, q) rational knots and links are formed by joining a twist containing p crossings to one containing q crossings, and then closing the knot/link up in a certain way. We note that the $(p, 2)$ rational knots are themselves a special family of knots called the **twist knots**.

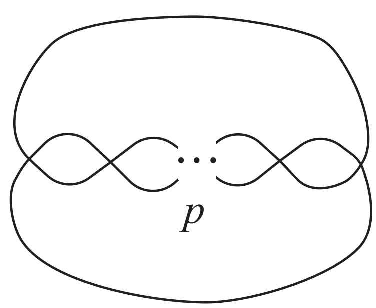

Figure 6: A $(2, p)$ torus knot/link.

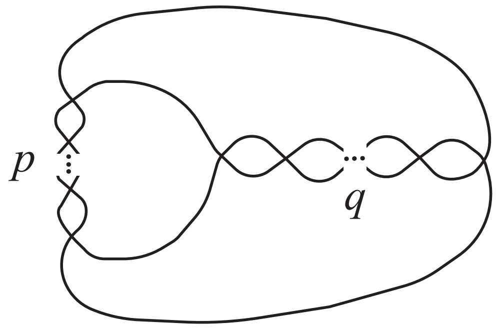

Figure 7: A (p, q) rational knot/link.

In Section 3, we will study the KnotLink game played on shadows of these families. But first, in Section 2, let us look at a sample game.

2 A Sample KnotLink Game

In this section, we provide a sample game, where the Simplifier and the Complicator compete in the KnotLink game on a twist knot diagram. Here, the Complicator moves first.

In Figure 8, we demonstrate the game. Diagram (a) shows our starting game board, and in diagram (b), the Complicator begins by resolving a crossing in the 5 twist. The Simplifier responds in diagram (c) by resolving an adjacent crossing in the 5 twist in such a way that a simplifying R2 move can be performed. Next, the Complicator moves in diagram (d) on a crossing in the 5 twist, performing a smoothing. The Simplifier then uses their one smoothing move in diagram (e) in the 2 twist. In the remaining moves, players take turns resolving the remaining crossings. Finally, a diagram of the unknot is produced and the Simplifier wins.

Could the Complicator have won this game if they had played more strategically? In the next section, we'll find out!

3 Results for Special Knot Families

Since the KnotLink game is a type of combinatorial game, given a game board and a choice of who plays first, there is a player who can guarantee themselves a win by playing following a certain strategy. We aim to determine who that player is—by identifying their winning strategy—for a number of game boards.

3.1 The $(2, p)$ torus knots and links

We begin our investigation of winning strategies with a simple result for $(2, p)$-torus knots.

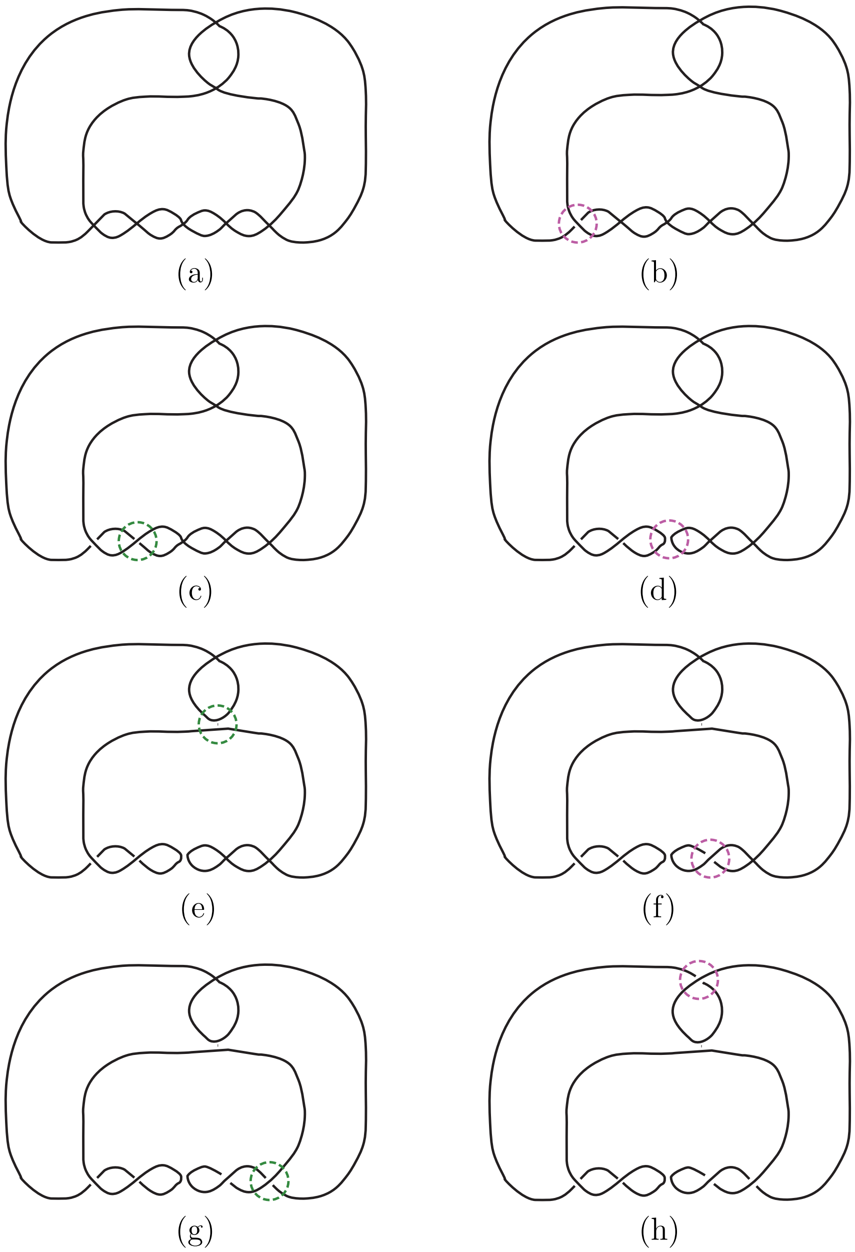

Figure 8: A sample game.

Proposition 3.1 *Consider a game board consisting of a standard* $(2, p)$*-torus knot/link diagram. The Simplifier has a winning strategy, both moving first and moving second.*

Proof. If the Simplifier moves first, then they should smooth at a crossing in such a way that a unknotted pseudodiagram is produced.

If the Complicator moves first, they will either resolve a crossing (preserving the general $(2, p)$-torus link structure) or else they will smooth at a crossing to create a simpler torus link or an unknotted pseudodiagram. The Complicator loses immediately if they make an unknotted pseudodiagram. In the case that they preserve the torus link structure, the Simplifier should smooth at a crossing in such a way that an unknotted pseudodiagram is produced. □

3.2 The (p, q) rational knots and links

Next, we'll turn our attention to (p, q) rational knots and links. We consider every type of (p, q) rational knot/link by considering every possible combination of parities (i.e., odd/even information) for p and q.

3.2.1 Both p and q are even

We begin with the following theorem for p and q both even.

Theorem 3.2 *If the KnotLink Game is played on a standard (p, q) rational knot diagram (see Figure 7) where both p and q are positive and even, then the second player has winning strategy regardless of their goal.*

As with the remaining theorems, we prove it by proving two lemmas. Each lemma specifies which player plays first. In most of the proofs in this section, we proceed by cases. When the first player is the losing player, these cases look like the diagrams given in Figure 9. Note that *which* crossing a player moves on within a twist does not affect game strategy or game outcomes, so without loss of generality, we move on the first crossing in each twist.

Lemma 3.3 *If the KnotLink Game is played on a standard (p, q) rational knot diagram where both p and q are even and the Complicator moves first, then the Simplifier has winning strategy.*

Proof.

Case 1: Suppose the Complicator performs a horizontal smoothing in the q twist. Then, the Simplifier should perform a vertical smoothing in the q twist. Now, neither player has the smoothing move available to them, and there are an even number of crossings in the p twist and the q twist remaining. For the rest of the game, when the Complicator moves in q, the Simplifier should move in q, resolving any crossing in any way since the q twist crossings no longer affect the linking of the diagram. Moreover, when the Complicator resolves a crossing in p, the Simplifier should respond by resolving an adjacent crossing in p so that an R2 move can be performed to remove the two crossings from the diagram. This will result in an unlink diagram.

Case 2: Suppose the Complicator performs a vertical smoothing in the q twist. Then, the Simplifier should perform a horizontal smoothing in the q twist and then follow the same strategy as in Case 1.

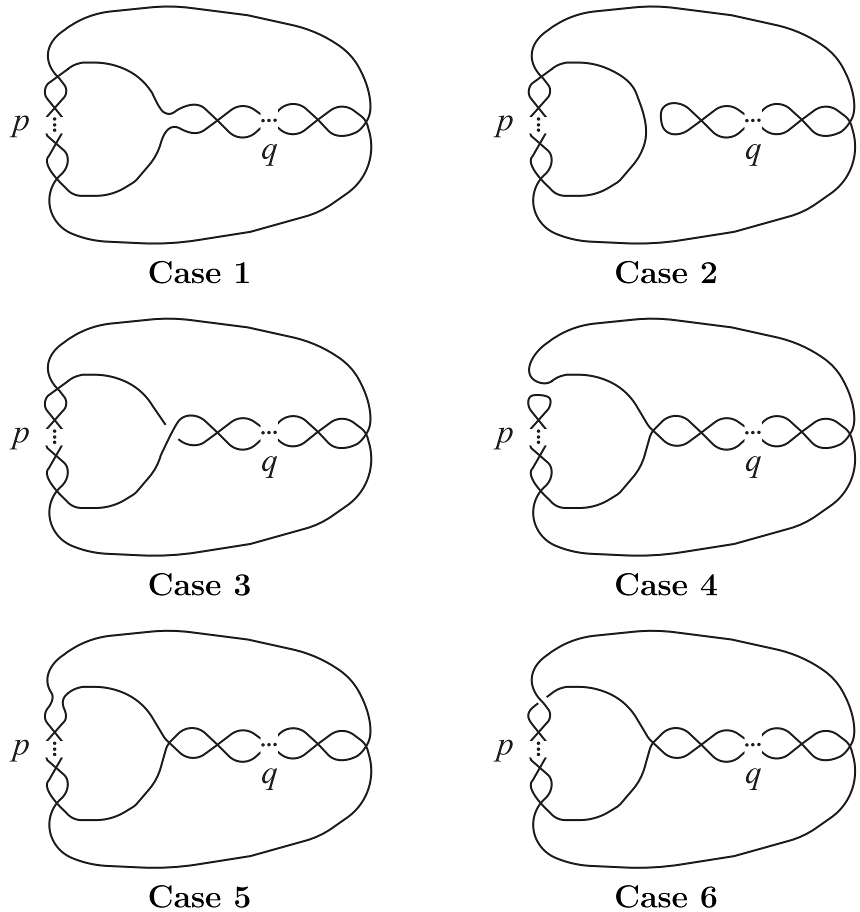

Figure 9: The six cases considered in the proofs of Lemmas 3.3 and 3.4.

Case 3: Suppose the Complicator resolves a crossing in the q twist. Then the Simplifier should resolve an adjacent crossing in q so that an R2 move can be performed, reducing the game to a game on a smaller, $(p, q-2)$ rational knot (if $q > 2$) or an unknotted pseudodiagram (if $q = 2$).

Case 4: Suppose the Complicator performs a horizontal smoothing in the p twist. This case is analogous to Case 2 and should be handled similarly.

Case 5: Suppose the Complicator performs a vertical smoothing in the p twist. This case is analogous to Case 1 and should be handled similarly.

Case 6: Finally, suppose the Complicator resolves a crossing in the p twist. This case is analogous to Case 3 and should be handled similarly.

$\square$

Lemma 3.4 *If the KnotLink Game is played on a standard (p, q) rational knot diagram where both p and q are even and the Simplifier moves first, then the Complicator has* *winning strategy.*

Proof.

Case 1: Suppose the Simplifier performs a horizontal smoothing in the q twist. Then, the Complicator should perform a vertical smoothing in the q twist. Now, neither player has the smoothing move available to them, and there are an even number of crossings in the p twist and the q twist remaining. For the rest of the game, when the Simplifier moves in q, the Complicator should move in q, resolving any crossing in any way since the q twist crossings no longer affect the linking of the diagram. Moreover, *until the very last move* by the Complicator, when the Simplifier resolves a crossing in p, the Complicator should respond by resolving an adjacent crossing in p so that an R2 move can be performed to remove the two crossings from the diagram. The *last move* by the Complicator in p, however, should be an alternating resolution of the final crossing to produce the Hopf link.

Case 2: Suppose the Simplifier performs a vertical smoothing in the q twist. Then, the Complicator should perform a horizontal smoothing in the q twist and then follow the same strategy as in Case 1.

Case 3: Suppose the Simplifier resolves a crossing in the q twist. If $q > 2$, then the Complicator should resolve an adjacent crossing in the q twist to enable an R2 move, reducing the game to a new game on a smaller, $(p, q-2)$ rational knot. If $q = 2$, then the Complicator should resolve the remaining q crossing so that the two q twist crossings are alternating. From here, if the Simplifier smooths in one direction in p, the Complicator should smooth another p crossing in the opposite direction, producing a pseudodiagram of the Hopf link. If the Simplifier resolves a crossing, the Complicator should resolve an adjacent crossing so that an R2 move can be performed *until the last move*, when the Complicator should make the remaining two crossings in p alternating, finally producing either the trefoil or the figure-eight knot.

Case 4: Suppose the Simplifier performs a horizontal smoothing in the p twist. This case is analogous to Case 2 and should be handled similarly.

Case 5: Suppose the Simplifier performs a vertical smoothing in the p twist. This case is analogous to Case 1 and should be handled similarly.

Case 6: Finally, suppose the Simplifier resolves a crossing in the p twist. This case is analogous to Case 3 and should be handled similarly.

□

3.2.2 The parities of p and q are different

Now we turn our attention to the especially tricky case of (p, q) rational knots where the parities of p and q are different from one another.

Theorem 3.5 *If the KnotLink Game is played on a standard (p, q) rational knot diagram where p is odd and q is even, then the Simplifier wins, regardless of whether they move first or second.*

To prove this theorem, we consider the cases where the Simplifier plays first and the Simplifier plays second in two lemmas.

Lemma 3.6 *If the Simplifier plays first on a standard (p, q) rational knot diagram where p is odd and q is even, then the Simplifier wins.*

Proof. The Simplifier's first move should be a vertical smoothing in the q twist. (See Figure 9, Case 2.) Now, the only crossings potentially affecting the nontriviality of the knot or link that will be produced by game play are the crossings in the p twist; each of the crossings remaining in q can either (a) eventually be undone with R1 moves, (b) produce a trivial, unlinked component if the Complicator smooths vertically in q or (c) disappear altogether, if the Complicator smooths horizontally in q.

Suppose the Complicator does not choose to smooth a crossing in p during game play. Then, since the number of crossings in the p twist is odd and the number of remaining crossings in the q twist is odd, regardless of where the Complicator moves, the Simplifier can play the remainder of the game so that they choose crossing information for exactly $\lfloor \frac{p}{2} \rfloor$ or $\lceil \frac{p}{2} \rceil$ crossings in p. Moreover, the Simplifier can resolve $\lfloor \frac{p}{2} \rfloor$ of these crossings *in response to* how the Complicator resolves a crossing in p, guaranteeing that $\lfloor \frac{p}{2} \rfloor$ R2 moves can be performed to eliminate all but one crossing from p. If p only has one crossing, the resulting diagram is guaranteed to be unknotted. Note that, this strategy implicitly assumes that if the Complicator moves in the q twist and there are still available moves in the q twist, the Simplifier will resolve a crossing in q.

Now, if the Complicator *did* decide to use their smoothing in the p twist at some point during game play, a horizontal smoothing would produce an unknotted pseudodiagram, handing the Simplifier a win. So for our final case, suppose the Complicator uses a vertical smoothing in the p twist at some point during game play. If the Simplifier has been following the resolution strategy above to be able to perform as many R2 moves as possible in p, at this point in the game, there are either no resolved crossings in the p twist or one resolved crossing (that can't be removed with an R2 move). If there is one resolved crossing in p, the Simplifier should resolve another crossing to enable an R2 move. If there are no resolved crossings in p (that can't be removed with R2 moves), then an even number of moves had been performed in p before the smoothing, guaranteeing that there are still an odd number of available moves in the q twist. So, the Simplifier should resolve a crossing in q next. Now, at this point in the game, there are an even number of unknown crossings in p and an even number of unknown crossings in q, and it is the Complicator's turn next. Now that the Complicator has used their smoothing, the game can proceed with the Simplifier responding to the Complicator's resolutions in p with resolutions in p that result in R2 simplifications and resolutions in q with resolutions in q. □

Lemma 3.7 *If the Complicator plays first on a standard* (p, q) *rational knot diagram where* p *is odd and* q *is even, then the Simplifier wins.*

Proof.

Case 1: Suppose the Complicator performs a horizontal smoothing in the q twist. According to the results of [3], if the Simplifier doesn't use their smoothing, the Complicator will have a winning strategy. Thus, the Simplifier should use their smoothing, performing a second horizontal smoothing in the q twist to produce an odd-even rational knot on which the remainder of the game is played essentially as a Knotting-Unknotting Game (since both players have used their smoothings). According to Theorem 13.3.2 in [7], the Simplifier/Unknotter has a winning strategy in this game.

Case 2: Suppose the Complicator performs a vertical smoothing in the q twist. The Simplifier should respond by performing a horizontal smoothing in the p twist to produce an unknotted pseudodiagram.

Case 3: Suppose the Complicator resolves a crossing in the q twist. The Simplifier should respond by resolving an adjacent crossing in the q twist so that an R2 move can be performed. This either reduces the game to a new game played on a smaller, $(p, q-2)$ rational knot, or if $q = 2$, this results in an unknotted pseudodiagram.

Case 4: Suppose the Complicator performs a horizontal smoothing in the p twist. In this case, the Simplifier should perform a vertical smoothing in the q twist, producing an unknotted pseudodiagram.

Case 5: Suppose the Complicator performs a vertical smoothing in the p twist. This produces an even-even rational knot. According to the results in [6], if the Simplifier doesn't use their smoothing move, the Complicator has a winning strategy. Thus, the Simplifier should smooth next. Performing another vertical smoothing in the p twist produces an odd-even rational knot on which the remainder of the game is played essentially as a Knotting-Unknotting Game (since both players have used their smoothings). According to Theorem 13.3.2 in [7], the Simplifier/Unknotter has a winning strategy in this game.

Case 6: Finally, suppose the Complicator resolves a crossing in the p twist. If $p > 1$, then the Simplifier should resolve an adjacent crossing in p so that an R2 move can be performed to reduce the game to a new game played on a smaller, $(p-2, q)$ rational knot. If $p = 1$, the Simplifier should perform a vertical smoothing in q to produce an unknotted pseudodiagram.

□

The following corollary can be easily proven by essentially rotating the above strategies by 90 degrees. In other words, if we swap all p's and q's in the proofs above and every instance of a horizontal smoothing is replaced by a vertical one and vice versa, the same proofs justify the following result.

Corollary 3.8 *If the KnotLink Game is played on a standard* (p, q) *rational knot diagram* *where p is even and q is odd, then the Simplifier wins, regardless of whether they move first or second.*

3.2.3 Both p and q are odd

Finally, we consider the case of (p, q) rational links where p and q are both odd.

Theorem 3.9 *If the KnotLink Game is played on a standard (p, q) rational link diagram where both p and q are odd, then the Simplifier wins, regardless of whether they move first or second.*

We prove our theorem with two lemmas.

Lemma 3.10 *If the Simplifier plays first on a standard (p, q) rational link diagram where p and q are both odd, then the Simplifier wins.*

Proof. Let's begin by considering the special case where $p = q = 1$. In this case, the Simplifier should smooth at any crossing in any direction to create an unknotted pseudodiagram. Otherwise, let the Simplifier's first move be a resolution of a crossing in the q twist. Now, we have six cases to consider for the Complicator's first move. Note that the first three cases only apply if $q > 1$.

Case 1: Suppose the Complicator performs a horizontal smoothing in the q twist. This transforms the game into one where p is odd and q is even, where the Simplifier has a winning strategy moving first or second. Since the Simplifier still has their smoothing move, the strategy of Theorem 3.5 guarantees the Simplifier a win in this case.

Case 2: Suppose the Complicator performs a vertical smoothing in the q twist. The Simplifier can now create an unknotted pseudodiagram by performing a horizontal smoothing in the p twist.

Case 3: Suppose the Complicator resolves a crossing in the q twist. The Simplifier should resolve another crossing in the q twist to reduce the diagram to a simpler odd-odd game.

Case 4: Suppose the Complicator performs a horizontal smoothing in the p twist. The Simplifier can now create an unknotted pseudodiagram by performing a vertical smoothing in the q twist.

Case 5: Suppose the Complicator performs a vertical smoothing in the p twist. If $q > 1$, then the Simplifier may resolve a second crossing in the q twist to allow an R2 move to be performed and reduce the game board to an even-odd starting board. Corollary 3.8 now guarantees the Simplifier a win since the Simplifier has not yet used their smoothing move. On the other hand, if $q = 1$ we have that $p > 1$, so the Simplifier can perform a horizontal smoothing in the p twist to produce an unknotted pseudodiagram.

Case 6: Finally, suppose the Complicator resolves a crossing in the p twist. Suppose $q > 1$. If $p > 1$, the Simplifier should resolve another crossing in the p twist so that an R2 move can be performed, reducing to a simpler game. If $p = 1$, then the Simplifier should perform a vertical smoothing in the q twist to produce an unknotted pseudodiagram. Now, if $q = 1$, regardless of what p is, the Simplifier should perform a horizontal smoothing in the p twist to produce an unknotted pseudodiagram.

□

Lemma 3.11 *If the Complicator plays first on a standard* (p, q) *rational link diagram where* p *and* q *are both odd, then the Simplifier wins.*

Proof. Since the losing player moves first, we consider all six possible opening moves that the Complicator could choose.

Case 1: Suppose the Complicator performs a horizontal smoothing in the q twist. Then the game board is equivalent to the starting game board described in Theorem 3.5. In the odd-even game, the Simplifier has a winning strategy. Since the Simplifier has not yet used their smoothing move in this game, the Simplifier has a winning strategy.

Case 2: Suppose the Complicator performs a vertical smoothing in the q twist. Then the Simplifier can immediately produce an unknotted pseudodiagram by performing a horizontal smoothing in the p twist.

Case 3: Suppose the Complicator resolves a crossing in the q twist. If $q > 1$, the Simplifier should resolve a crossing so that an R2 move can be performed to simplify the game board to a smaller odd-odd starting configuration. It is as if we are beginning a new game since no player has used their smoothing move. On the other hand, if $q = 1$, the Simplifier should do a horizontal smoothing in p to produce an unknotted pseudodiagram.

Case 4: Suppose the Complicator performs a horizontal smoothing in the p twist. Then the Simplifier can immediately produce an unknotted pseudodiagram by performing a vertical smoothing in the p twist, just as in Case 2.

Case 5: Suppose the Complicator performs a vertical smoothing in the p twist. Then, just as in Case 1, our previous results (Corollary 3.8) guarantee that the Simplifier has a winning strategy.

Case 6: Finally, suppose the Complicator resolves a crossing in the p twist. If $p > 1$, then the Simplifier should resolve a neighboring crossing in such a way that an R2 move can be performed. This reduces the game board to a smaller odd-odd game. If $p = 1$, then the Simplifier can produce an unknotted pseudodiagram by performing a vertical smoothing in q.

Since every case provides the Simplifier a winning strategy, our proof is complete. □

3.2.4 A corollary for twist knots

As we noted in Section 1, twist knots are a special subfamily of rational knots, the $(p, 2)$ rational knots. Our results from Subsections 3.2.1 and 3.2.2 can be applied to this family to give us the following corollary.

Corollary 3.12 *Suppose the KnotLink Game is played on a standard diagram of a twist knot with $p + 2$ crossings. Then the Complicator has a winning strategy if and only if p is even and the Simplifier moves first.*

So, for example, in our sample game, our two players played on a twist knot with $p = 5$. Since p is odd in this game, the Simplifier has a winning strategy regardless of if they move first or second. So, unfortunately, the Complicator couldn't have done any better in this game if the Simplifier is playing well.

4 Conclusion

In the previous section, we proved results for two knot and link families, the $(2, p)$-torus knots/links and the (p, q) rational knots/links. In fact, the $(2, p)$-torus knots/links are rational knots and links given by the 1-tuple (p), so we have found winning strategies for rational knot and link families given by $(p_1, p_2, ..., p_n)$ for $n = 1$ and $n = 2$. The next natural question is: what about $n = 3$? Who has a winning strategy when playing on rational knots and links of the form (p, q, r) for various choices of parity of p, q, and r and a choice of which player moves first? A more ambitious team of researchers might ask: what about for an arbitrary n? Can a characterization be given for when the Simplifier has a winning strategy and when the Complicator has a winning strategy?

Of course, there are many more knot and link shadows (infinitely many!) that don't look like these standard rational knot and link shadows. Each shadow represents an interesting open question. Given this abundance of open questions, we invite others to play with us!

Acknowledgments

The authors would also like to thank the Simons Foundation (#426566, Allison Henrich) for their support of this research.

References

[1] J.W. Alexander, G.B. Briggs, On types of knotted curves, *Ann. of Math. (2)*, **28** (1926/27), 562–586.

[2] S. Brown, F. Cabrera, R. Evans, G. Gibbs, A. Henrich, J. Kreinbihl, The region unknotting game, *Math. Mag.*, **90** (2017), 323–337.

[3] A. Giambrone, J. Murphy, The linking-unlinking game, *Involve*, **12** (2019), 1109–1141.

[4] A. Henrich, I. Johnson, The link smoothing game, *AKCE Int. J. Graphs Comb.*, **9** (2012), 145–159.

[5] A. Henrich, I. Johnson, J. Ostroff, The region smoothing swap game, *arXiv preprint, arXiv:1909.12370*, (2019), available online at the URL:`https://arxiv.org/abs/1909.12370`.

[6] A. Henrich, N. MacNaughton, S. Narayan, O. Pechenik, R. Silversmith, J. Townsend, A midsummer knot's dream, *College Math. J.*, **42** (2011), 126–134.

[7] W. Johnson, Combinatorial game theory, well-tempered scoring games, and a knot game, *arXiv:1107.5092 [math.CO]*, (2011), 1–279, available online at the URL:`https://arxiv.org/abs/1107.5092`.

[8] K. Reidemeister, Elementare Begründung der Knotentheorie, (German), *Abh. Math. Sem. Univ. Hamburg*, **5** (1927), 24–32.

Hunter Adams
Seattle University
901 12th Ave.
Seattle, WA 98122
E-mail: `adamsh3@seattleu.edu`

Allison Henrich
Seattle University
901 12th Ave.
Seattle, WA 98122
E-mail: `henricha@seattleu.edu`

Solden Stoll
O'Dea High School
802 Terry Ave.
Seattle, WA 98104
E-mail: `sstoll@seattleu.edu`

Received: May 30, 2020 **Accepted**: June 14, 2020
Communicated by Carmen Caprau