Minimal Generating Sets of the Monoid of Partial Order-Preserving Injections

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Abstract - Monoids arise in such fields as computer science, physics, and numerous branches of mathematics including abstract algebra, cryptography and operator theory. In this research project we seek to determine minimal generating sets for the monoid of partial order-preserving injections of an n-element set, $POI(n)$. A generating set for a monoid is a collection of elements S such that every element of the monoid can be expressed as a product of elements from S. Generating sets are of fundamental importance across math and science, and mathematicians have great interest in studying generating sets of a variety of algebraic structures. By a minimal generating set, we refer to a generating set for which no proper subset is a generating set. In this paper, we provide necessary and sufficient conditions for a set to be a minimal generating set for $POI(n)$, and we show that there are exactly $(n-1)!$ minimal generating sets for $POI(n)$.

Keywords : monoid; partial order-preserving injection; rank; minimal generating set; partial identity; Green's relations; directed graph; walk; path; cycle

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1 Introduction

Partial order-preserving injections comprise an important subset of the collection of all partial injections from a set to itself. This latter collection forms a monoid (defined below) into which any inverse monoid can be embedded [5]. Embeddings preserve algebraic structure, so that the collection of all partial injections on a set becomes a central object of study in the theory of inverse monoids. In this paper, we will be specifically studying the monoid consisting of partial order-preserving injections. These order-preserving maps often arise in the context of studying transformations of partially ordered sets. With this motivational background, let us turn to a brief review of basic concepts in the theory of monoids.

Definition 1.1 *A* ***monoid*** *is a non-empty set M together with a binary operation $*$ that satisfies the following axioms:*

1) Closure: If $a, b \in M$, then $a * b \in M$.

2) Associativity: $a * (b * c) = (a * b) * c$, for all a, b, and $c \in M$.

*3) Existence of an Identity: An element $e \in M$ is called the **identity element** if $a * e = a = e * a$ for every $a \in M$.*

Example 1.2 The set of positive real numbers under multiplication, $(\mathbb{R}^+, \times)$, is a classic example of a monoid, with identity element 1.

Example 1.3 Note $(\mathbb{R}^+, +)$ is not a monoid since 0 does not belong to $\mathbb{R}^+$. However, $([0, \infty), +)$ is a monoid with identity element 0.

Note that the monoid in Example 1.3 forms a **submonoid** of $(\mathbb{R}, +)$, since it is a subset of $\mathbb{R}$ that forms a monoid under $+$.

Definition 1.4 *A **semigroup** is a non-empty set S that satisfies the closure and associativity axioms in Definition 1.1. Thus, every monoid is a semigroup.*

Any semigroup can be made into a monoid by adjoining an identity element. In Example 1.3, for instance, starting with the semigroup $(\mathbb{R}^+, +)$, we form the monoid $([0, \infty), +) = (\mathbb{R}^+ \cup \{0\}, +)$.

Example 1.5 Let X be a non-empty set. An expression of the form $x_1 x_2 x_3 \cdots x_n$ (with $x_i \in X$) is called a **word** on the alphabet X. The set of all words on X is the **free semigroup** on X, denoted X^+. We can form a monoid by adjoining an empty word, say Λ, to X^+. This monoid, denoted X^*, is called the **free monoid** on X.

The operation in X^+ and X^* is concatenation. If we let X be the English alphabet of lowercase letters, then some elements of X^+ are

$$\begin{aligned} road \ * \ runner &= roadrunner, \\ aababba \ * \ babbba &= aababbababbba. \end{aligned}$$

Note that we often omit the operation $*$ from the notation when the operation is known. Thus, for instance, we may write $a * b$ simply as ab.

Example 1.6 The **symmetric inverse monoid**, ***SIM(S)***, consists of all partial injections from a set S to itself. The operation on $SIM(S)$ is given by composition of mappings, where we will adopt the convention that mappings are composed from left to right.

The monoid $SIM(S)$ is of great relevance to this paper. Therefore, let us take some time to elaborate on its structure. Consider the particular case where $S = \{1, 2, 3, \ldots, n\}$, denoted as $SIM(n)$. The identity element of *SIM(n)* is the element that fixes each element of the set $\{1, 2, 3, \ldots, n\}$. The elements of *SIM(n)* can be represented by a classic two-line notation. For example,

$$\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 4 & 6 & - & 5 \end{pmatrix} \in SIM(6). \tag{1}$$

In this element, we observe that 1 maps to 2, 2 maps to 1, 3 maps to 4, 4 maps to 6, 5 is not in the domain, and 6 maps to 5. We denote the domain of τ by $Dom(\tau)$ and the range of τ by $Rng(\tau)$.

Definition 1.7 *The **rank** of an element $\tau \in SIM(S)$ is*

$$rank(\tau) = |Dom(\tau)| = |Rng(\tau)|.$$

In (1), $rank(\tau) = 5$. Also, note that the set of all elements in $SIM(n)$ of rank n forms the well-known group called the symmetric group, denoted S_n. Consult any introductory text on modern algebra for more information([6], [8]).
To illustrate how composition of partial permutations works, consider this example:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 3 & 4 & - & 5 & 6 & - \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 2 & 1 & 4 & 6 & - & 5 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 4 & 6 & - & - & 5 & - \end{pmatrix}.$$

Of course, the rank of the composition of two partial mappings cannot exceed the rank of either element comprising the product. Here, for example, we see that the elements on the left side have rank 4 and 5, respectively, while the resulting product on the right side only has rank 3.

There are some fundamental equivalence relations which help us understand the structure of semigroups and monoids. These are known as the **Green's relations** [5]. To discuss these, we need some notation: given a monoid M and $t \in M$, denote $Mt = \{mt : m \in M\}$ and $tM = \{tm : m \in M\}$. Now, let $\alpha, \beta \in M$. Then

1. α and β are **$\mathcal{L}$-related**, denoted $\alpha\ \mathcal{L}\ \beta$, if and only if there exist $x, y \in M$ such that $x\alpha = \beta,\ y\beta = \alpha$. That is, $\beta \in M\alpha$ and $\alpha \in M\beta$.

2. α and β are **$\mathcal{R}$-related**, denoted $\alpha\ \mathcal{R}\ \beta$, if and only if there exist $u, v \in M$ such that $\alpha u = \beta,\ \beta v = \alpha$. That is, $\beta \in \alpha M$ and $\alpha \in \beta M$.

3. The **$\mathcal{D}$-relation**, $\mathcal{D} = \mathcal{L} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{L}$, is the smallest equivalence relation containing both $\mathcal{L}$ and $\mathcal{R}$.

4. The **$\mathcal{H}$-relation** is the intersection of $\mathcal{L}$ and $\mathcal{R}$. That is, α and β are $\mathcal{H}$-related, denoted $\alpha\ \mathcal{H}\ \beta$, if and only if $\alpha\ \mathcal{L}\ \beta$ and $\alpha\ \mathcal{R}\ \beta$.

The Green's relations are equivalence relations, and hence form equivalence classes, which we will refer to as $\mathcal{L}$-classes, $\mathcal{R}$-classes, $\mathcal{D}$-classes, and $\mathcal{H}$-classes.

We are now ready to begin studying the particular monoid of interest to us in our research, and in the next section, we will determine its Green's relations.

2 Partial Order-preserving Injections

The monoid of interest to our research is the monoid of **partial order-preserving injections**, denoted by ***POI(n)***, which is a submonoid of *SIM(n)*. An element σ of $SIM(n)$ is an element of $POI(n)$ if whenever $i < j$ in $\{1, 2, 3, \ldots, n\}$, then $\sigma(i) < \sigma(j)$. For example,

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & - & 5 & 6 & - \end{pmatrix} \in POI(6),$$

since the values on the second row appear in increasing order. However, the element τ in (1) is not an element of $POI(6)$.

Definition 2.1 *A* ***partial identity*** *in $POI(n)$ is an element σ such that for every integer i with $1 \leq i \leq n$, we have either $\sigma(i) = i$ or $i \notin Dom(\sigma)$.*

As we will see in our main results, partial identities play an important role in describing minimal generating sets of $POI(n)$. An example of a partial identity is the element

$$\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & - & 3 & - & 5 & - \end{pmatrix} \in POI(6),$$

whose rank is 3. Throughout this paper, we denote the subset of $POI(n)$ of elements of rank k as $POI_k(n)$. For example, $rank(\tau) = 3$, so $\tau \in POI_3(6)$. Note that

$$POI(n) = \bigcup_{k=0}^{n} POI_k(n),$$

and thus,

$$|POI(n)| = \sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n},$$

where the first equality can be found in [1] and the second is a well-known combinatorial formula that can be proven either by a routine application of induction or by a combinatorial argument. Note that the k^{th} term of the summation counts the number of elements in $POI_k(n)$. For example,

$$|POI(4)| = \sum_{k=0}^{4} \binom{4}{k}^2 = \binom{4}{0}^2 + \binom{4}{1}^2 + \binom{4}{2}^2 + \binom{4}{3}^2 + \binom{4}{4}^2 = \binom{8}{4} = 70.$$

We now look at the Green's relations for the monoid $POI(n)$. The next lemma summarizes the results. The proof is routine and is omitted.

Lemma 2.2 *Let $\alpha, \beta \in POI(n)$. Then:*

- $\alpha \,\mathcal{R}\, \beta$ *if and only if* $Dom(\alpha) = Dom(\beta)$.
- $\alpha \,\mathcal{L}\, \beta$ *if and only if* $Rng(\alpha) = Rng(\beta)$.
- $\alpha \,\mathcal{D}\, \beta$ *if and only if* $rank(\alpha) = rank(\beta)$.
- $\alpha \,\mathcal{H}\, \beta$ *if and only if* $\alpha = \beta$.

We give some examples of the $\mathcal{R}$ and $\mathcal{L}$ relations in $POI(n)$.

Example 2.3 Let

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 4 & 5 & 6 & 7 & - & - \end{pmatrix} \qquad \text{and} \qquad \beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 3 & 4 & 5 & 6 & - & - \end{pmatrix}$$

in $POI(7)$. Since

$$\alpha \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ - & 2 & - & 3 & 4 & 5 & 6 \end{pmatrix} = \beta$$

and

$$\beta \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ - & 2 & 4 & 5 & 6 & 7 & - \end{pmatrix} = \alpha,$$

we see that $\alpha \,\mathcal{R}\, \beta$.

Example 2.4 Let

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 4 & 5 & 6 & - & - & 7 & 8 \end{pmatrix} \qquad \text{and} \qquad \gamma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & - & 4 & 5 & 6 & 7 & 8 & - \end{pmatrix}$$

in $POI(8)$. We have

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & - & 2 & 3 & 4 & 7 & 8 & - \end{pmatrix} \sigma = \gamma$$

and

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 3 & 4 & 5 & - & - & 6 & 7 \end{pmatrix} \gamma = \sigma,$$

so we conclude that $\sigma \,\mathcal{L}\, \gamma$.

Remark 1. Let $k \in \{1, 2, \ldots, n\}$, and let $\alpha, \beta \in POI_k(n)$. Then $\alpha\beta \in POI_k(n)$ if and only if $Rng(\alpha) = Dom(\beta)$. In this case, $\alpha \,\mathcal{R}\, \alpha\beta$ and $\beta \,\mathcal{L}\, \alpha\beta$.

We will now turn our attention to the main problem in this paper, the identification of minimal generating sets for $POI(n)$.

Definition 2.5 *A* ***generating set*** *for an algebraic structure Z (such as a group, semigroup, monoid, vector space, and so on) is a collection $\mathcal{C}$ of elements in Z such that each element in Z can be expressed by applying a finite sequence of algebraic operations to elements in $\mathcal{C}$.*

In linear algebra, for example, a generating set for a vector space is commonly known as a spanning set.

Definition 2.6 *A generating set for Z is called an* ***minimal generating set*** *if no proper subset of it generates Z.*

The search for minimal generating sets of algebraic structures has led to a steady stream of work and has important applications in mathematics and computer science. In [2], these sets were classified for $SIM(n)$, while in [3], they were studied in the symmetric and alternating groups.

Returning to our study of $POI(n)$, observe that $POI_n(n)$ consists only of the identity element

$$id = \begin{pmatrix} 1 & 2 & 3 & \cdots & n-1 & n \\ 1 & 2 & 3 & \cdots & n-1 & n \end{pmatrix}.$$

This element is uninteresting with respect to the generating process, so we will focus our attention on generating the set $POI(n) - \{id\}$. For the remainder of this paper, when we speak of generating $POI(n)$, we will mean that we are generating $POI(n) - \{id\}$, which itself is a subsemigroup of $POI(n)$. The key to generating $POI(n) - \{id\}$ is to generate all of the rank $n-1$ elements first. In fact, it is customary to restrict generating sets of $POI(n)$ to consist of elements of rank $n-1$. By multiplying elements of $POI_{n-1}(n)$ together, we can obtain elements of lower rank (see Remark 1). We arrange the elements in $POI_{n-1}(n)$ in an $n \times n$ array as shown in Figure 1 for $POI(4)$. Elements in the same row have the same domain, elements in the same column have the same range, and elements on the main diagonal are partial identities. Let

$$\widehat{S}_i = \{1, 2, \ldots, i-1, i+1, \ldots, n-1, n\}$$

for $1 \leq i \leq n$. For example, for $n = 4$, if

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & - & 4 \end{pmatrix}, \tag{2}$$

then $Dom(\sigma) = \widehat{S}_3$ and $Rng(\sigma) = \widehat{S}_1$.

$$
\begin{array}{cccccc}
 & \text{Column 1} & \text{Column 2} & \text{Column 3} & \text{Column 4} & \\
\text{Row 1} & \begin{pmatrix}1&2&3&4\\1&2&3&-\end{pmatrix} & \begin{pmatrix}1&2&3&4\\1&2&4&-\end{pmatrix} & \begin{pmatrix}1&2&3&4\\1&3&4&-\end{pmatrix} & \begin{pmatrix}1&2&3&4\\2&3&4&-\end{pmatrix} & \hat{S}_4 \\
\text{Row 2} & \begin{pmatrix}1&2&3&4\\1&2&-&3\end{pmatrix} & \begin{pmatrix}1&2&3&4\\1&2&-&4\end{pmatrix} & \begin{pmatrix}1&2&3&4\\1&3&-&4\end{pmatrix} & \begin{pmatrix}1&2&3&4\\2&3&-&4\end{pmatrix} & \hat{S}_3 \\
\text{Row 3} & \begin{pmatrix}1&2&3&4\\1&-&2&3\end{pmatrix} & \begin{pmatrix}1&2&3&4\\1&-&2&4\end{pmatrix} & \begin{pmatrix}1&2&3&4\\1&-&3&4\end{pmatrix} & \begin{pmatrix}1&2&3&4\\2&-&3&4\end{pmatrix} & \hat{S}_2 \\
\text{Row 4} & \begin{pmatrix}1&2&3&4\\-&1&2&3\end{pmatrix} & \begin{pmatrix}1&2&3&4\\-&1&2&4\end{pmatrix} & \begin{pmatrix}1&2&3&4\\-&1&3&4\end{pmatrix} & \begin{pmatrix}1&2&3&4\\-&2&3&4\end{pmatrix} & \hat{S}_1 \\
 & \hat{S}_4 & \hat{S}_3 & \hat{S}_2 & \hat{S}_1 &
\end{array}
$$

Figure 1: 4×4 array of the elements in $POI_3(4)$

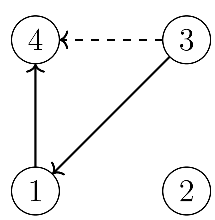
Note that if σ is an element of $POI_{n-1}(n)$ with $Dom(\sigma) = \widehat{S}_i$ and $Rng(\sigma) = \widehat{S}_j$, then we can uniquely denote σ as the pair of numbers (i, j). For example, the element σ in (2) can now be represented as $\sigma = (3, 1)$.

Notation: Let X be a subset of $POI_{n-1}(n)$. Let Ω_X denote a directed graph on $\{1, 2, \ldots, n\}$, where $\mathbf{a} \curvearrowright \mathbf{b}$ is a directed edge in Ω_X if and only if $(a, b) \in X$.
If $\sigma = (a, b)$ and $\tau = (c, d)$ are two elements of $POI_{n-1}(n)$, then note that $\sigma\tau = (a, b)(c, d) = (a, d)$ if $b = c$. If $b \neq c$, by Remark 1 the rank of $\sigma\tau$ drops and we do not express $\sigma\tau$ in this ordered pair notation. Multiplication can now be performed by following directed edges, as shown in our next examples.

Example 2.7 In $POI(4)$,

$$\begin{pmatrix}1&2&3&4\\2&3&-&4\end{pmatrix}\begin{pmatrix}1&2&3&4\\-&1&2&3\end{pmatrix} = \begin{pmatrix}1&2&3&4\\1&2&-&3\end{pmatrix}$$

can be expressed as $(3, 1)(1, 4) = (3, 4)$, which is represented graphically as



(3)

In (3), the dotted arrow indicates the result of the composition of the two solid arrows.

Example 2.8 Again in $POI(4)$,

$$\begin{pmatrix}1&2&3&4\\-&1&2&3\end{pmatrix}\begin{pmatrix}1&2&3&4\\1&2&4&-\end{pmatrix}\begin{pmatrix}1&2&3&4\\1&3&-&4\end{pmatrix} = \begin{pmatrix}1&2&3&4\\-&1&3&4\end{pmatrix}$$

can be represented as $(1, 4)(4, 3)(3, 2) = (1, 2)$, which is represented graphically as

$$\begin{tikzcd} 4 \arrow[r] & 3 \arrow[d] \\ 1 \arrow[u] \arrow[r, dashed] & 2 \end{tikzcd} \tag{4}$$

In (3) and (4), the solid directed edges form an example of a *walk* [4].

Definition 2.9 *A* ***walk*** *is a sequence of* m *edges of the form*

$$x_0 \to x_1 \to x_2 \to \cdots \to x_m,$$

where $x_0, x_1, x_2, \ldots, x_m$ *are (not necessarily distinct) vertices, and* $x_i \to x_j$ *indicates that there is an edge from* x_i *to* x_j*. If* $x_0 = x_m$*, then the walk is* ***closed****.*

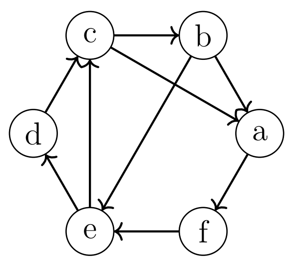


Figure 2: An example of a directed graph

For example, in Figure 2 we have a walk $c \to b \to e \to d \to c \to a$. Note that if we remove the edge $c \to a$, we have a closed walk: $c \to b \to e \to d \to c$.

Definition 2.10 *If a walk is composed of distinct edges and has distinct vertices, then it is called a* ***path****. That is, in a path we have* $x_i \neq x_j$ *for* $0 \leq i < j \leq m$*, except possibly for* $x_0 = x_m$*. In the case where* $x_0 = x_m$*, the path is closed and usually referred to as a* ***cycle*** *or* ***m-cycle****.*

In Figure 2, we have the path $e \to d \to c \to b$ and the 4-cycle $e \to d \to c \to b \to e$.

Shortly, we will see how this basic graph theory can help us describe minimal generating sets of *POI(n)*. Our results were first motivated by studying output results from a program written using the programming language Groups, Algorithms, and Programming (GAP) [7]. The program explicitly computed all possible generating sets for *POI(n)* for small integers n. Let us now describe the main results we obtained.

3 Main Theorems

Our first result provides a necessary condition for a subset $X \subseteq POI_{n-1}(n)$ to be a minimal generating set for $POI(n)$.

Lemma 3.1 *Let X be a subset of $POI_{n-1}(n)$ consisting of n elements that generate $POI(n)$. Then X contains (a) no two distinct $\mathcal{R}$-related elements, (b) no two distinct $\mathcal{L}$-related elements, and (c) no partial identities.*

Proof.

We prove all three parts by way of contradiction. First, to prove part (a), we assume that there are two $\mathcal{R}$-related elements in X. Using Lemma 2.2, we deduce that for some $i \in \{1, 2, \ldots, n\}$, the subset $\widehat{S}_i$ is the domain for no elements in X. Let $\gamma \in POI_{n-1}(n)$ be any element such that $Dom(\gamma) = \widehat{S}_i$. Write $\gamma = (i, b)$. Since X is a generating set, then there exist $\xi_1, \xi_2, \ldots, \xi_k \in X$ such that

$$\gamma = \xi_1 \xi_2 \cdots \xi_k.$$

Clearly, each ξ_j must belong to $POI_{n-1}(n)$. Therefore, writing $\xi_j = (x_{j1}, x_{j2})$ for each j with $1 \leq j \leq k$, we have

$$\gamma = (x_{11}, x_{12})(x_{21}, x_{22}) \cdots (x_{k1}, x_{k2}) = (i, b),$$

where $x_{j2} = x_{(j+1)1}$ for each j with $1 \leq j < k$. It follows that $i = x_{11}$, and thus $Dom((x_{11}, x_{12})) = \widehat{S}_i$, which is a contradiction.

Similarly, for part (b), assume X contains two $\mathcal{L}$-related elements. Again by Lemma 2.2, there is some subset $\widehat{S}_i$ that no element in X has as its range. Let $\alpha \in POI_{n-1}(n)$ be any element such that $Rng(\alpha) = \widehat{S}_i$. Write $\alpha = (a, i)$. Then there exist $\xi_1, \xi_2, \ldots, \xi_k \in X$ such that $\alpha = \xi_1 \xi_2 \cdots \xi_k$. Writing $\xi_j = (x_{j1}, x_{j2})$ as before, we have

$$\alpha = (x_{11}, x_{12})(x_{21}, x_{22}) \cdots (x_{k1}, x_{k2}) = (a, i),$$

where $x_{j2} = x_{(j+1)1}$ for each $1 \leq j < k$. It follows that $i = x_{k2}$, which implies $Rng((x_{k1}, x_{k2})) = \widehat{S}_i$, which is a contradiction.

Finally, we prove part (c). Assume X contains a partial identity element, say $\omega = (a, a)$, which fixes the elements of $\widehat{S}_a$. Choose any element $\gamma = (a, b) \in POI_{n-1}(n)$ with $\gamma \neq \omega$. Then, following the notation above, γ can be expressed as

$$\gamma = \xi_1 \xi_2 \cdots \xi_k = (x_{11}, x_{12})(x_{21}, x_{22}) \cdots (x_{k1}, x_{k2}) = (a, b).$$

As above, we get $a = x_{11}$. By part (a), the only element of X of the form (a, y) is the partial identity (a, a). Therefore, it must be the case that $(x_{11}, x_{12}) = \omega$. Hence,

$$\gamma = (a, a)(x_{21}, x_{22}) \cdots (x_{k1}, x_{k2}).$$

Now we deduce that $x_{21} = a$, and we can repeat the same reasoning as above to conclude that $x_{22} = a$. Therefore,

$$\gamma = (a, a)(a, a) \cdots (x_{k1}, x_{k2}).$$

Proceeding in this way, we eventually have

$$\gamma = (a, a)(a, a) \cdots (a, a) = (a, a) = \omega,$$

a contradiction. $\square$

As a consequence of the proof of Lemma 3.1, observe that every $\widehat{S}_i$ (for $1 \leq i \leq n$) must be the domain of some generator in a generating set $POI(n)$. This therefore proves the following:

Corollary 3.2 *Every generating set of $POI(n)$ must contain at least n elements.*

We are now ready to state and prove the central result of this paper.

Theorem 3.3 *Let X be a subset of n elements of $POI_{n-1}(n)$ with no two distinct $\mathcal{R}$-related elements, no two distinct $\mathcal{L}$-related elements, and no partial identities. Then X is a minimal generating set for $POI(n)$ if and only if the edges of Ω_X form an n-cycle.*

Proof. First, assume X is a minimal generating set for $POI(n)$. The assumption that X has no partial identities and that no two distinct members of X are R-related or L-related implies that no vertex in Ω_X can have in-degree or out-degree exceeding 1. As a result, Ω_X is a union of disjoint directed cycles. If we assume by way of contradiction that the edges of Ω_X do not form an n-cycle, this implies that we must have at least two smaller cycles, say C_1 and C_2, of lengths n_1 and n_2, respectively. Let us choose an element $(a, b) \in POI_{n-1}(n)$, where a is a vertex of the n_1-cycle C_1 and b is a vertex of the n_2-cycle C_2. Since X is a generating set, we can express (a, b) as a product of generators:

$$(a, b) = (x_{11}, x_{12})(x_{21}, x_{22}) \cdots (x_{k1}, x_{k2}).$$

However, since $x_{11} = a$ is a vertex in C_1, then x_{12} is also a vertex in C_1, which implies (x_{11}, x_{12}) is a directed edge in $\Omega_X - C_2$. Similarly, since $x_{12} = x_{21}$ is a vertex in C_1 then $x_{22} = x_{31}$ is a vertex in C_1 which implies (x_{21}, x_{22}) is a directed edge in $\Omega_X - C_2$. Continuing this process, we find that (x_{k1}, x_{k2}) is a directed edge in $\Omega_X - C_2$, which is a contradiction to the fact that $x_{k2} = b$ is a vertex in C_2. Hence, Ω_X forms an n-cycle.

Now assume that the edges of Ω_X form an n-cycle. To show that X is a generating set, let (a, b) be an arbitrary element in $POI_{n-1}(n)$. Since the edges of Ω_X form an n-cycle, then we have a cycle of the form

$$x_1 \to x_2 \to \cdots \to x_{i-1} \to a \to x_{i+1} \to \cdots \to x_{j-1} \to b \to x_{j+1} \to \cdots \to x_n \to x_1.$$

Clearly, there is a path from a to b; namely,

$$a \to x_{i+1} \to \cdots \to x_{j-1} \to b.$$

Therefore, (a, b) can be generated by X as

$$(a, b) = (a, x_{i+1})(x_{i+1}, x_{i+2}) \cdots (x_{j-1}, b).$$

Once all the elements of $POI_{n-1}(n) - \{id\}$ are generated, there is a straightforward algorithm for generating all of the elements of $POI_k(n)$ for $k \leq n - 2$. We describe this algorithm here and Example 3.4 following the proof will illustrate it.

First, it is easy to generate all partial identity elements of all ranks by simply multiplying together the appropriate partial identities of rank $n - 1$. Now consider any element $\sigma \in POI_k(n)$ for $k \leq n - 2$. Let $\tau \in POI_k(n)$ be a partial identity element with $Dom(\tau) = Dom(\sigma)$. Let $Rng(\sigma) = \{a_1, a_2, \ldots, a_k\}$ and $Rng(\tau) = \{b_1, b_2, \ldots, b_k\}$ with $a_1 < a_2 < \cdots < a_k$ and $b_1 < b_2 < \cdots < b_k$. We now conduct a series of steps that correspond to right-multiplication of τ by elements of rank $n - 1$ to replace the k-tuple $[b_1, b_2, \ldots, b_k]$ with $[a_1, a_2, \ldots, a_k]$ in the range. This is accomplished by changing exactly one entry of the k-tuple at a time by exactly one number in such a way that the increasing ordering of the entries of the k-tuple is maintained. To change b_i to $b_i \pm 1$, we use the element of rank $n-1$ denoted in the pair notation $\gamma = (b_i \pm 1, b_i)$. A direct calculation verifies that $Dom(\tau\gamma) = Dom(\tau)$ and $Rng(\tau\gamma) = \{b_1, b_2, \ldots, b_{i-1}, b_i \pm 1, b_{i+1}, \ldots, b_k\}$. Therefore, we can construct a finite series of elements $\gamma_1, \gamma_2, \ldots, \gamma_r$ belonging to $POI_{n-1}(n)$ such that

$$Dom(\tau\gamma_1\gamma_2 \cdots \gamma_r) = Dom(\tau) = Dom(\sigma)$$

and

$$Rng(\tau\gamma_1\gamma_2 \cdots \gamma_r) = Rng(\sigma).$$

This implies that

$$\sigma = \tau\gamma_1\gamma_2 \cdots \gamma_r. \tag{5}$$

Since τ is a partial identity and each γ_i has rank $n - 1$, we can see from (5) that σ is generated by elements in $POI_{n-1}(n)$. This proves that X is a generating set for $POI(n)$, and since X contains exactly n elements by assumption, Corollary 3.2 implies that X is a minimal generating set. □

Example 3.4 Let $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ - & - & 1 & 5 & 6 & - \end{pmatrix} \in POI(6)$. Here, we start with the partial identity $\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ - & - & 3 & 4 & 5 & - \end{pmatrix}$. Clearly, $Dom(\tau) = Dom(\sigma)$. We now carry out the series of steps described in the algorithm in the proof of Theorem 3.3 to replace $[3, 4, 5]$ with $[1, 5, 6]$. One way to do this is via the replacements

$$[3, 4, 5] \longrightarrow [2, 4, 5] \longrightarrow [1, 4, 5] \longrightarrow [1, 4, 6] \longrightarrow [1, 5, 6].$$

According to the algorithm, these four steps can be accomplished by right-multiplication by the elements

$$\gamma_1 = (2, 3), \qquad \gamma_2 = (1, 2), \qquad \gamma_3 = (6, 5), \qquad \gamma_4 = (5, 4).$$

We can easily compute that indeed we have

$$\tau\gamma_1\gamma_2\gamma_3\gamma_4 = \sigma.$$

Let us now illustrate some examples of Theorem 3.3.

Example 3.5 The set

$$X = \left\{ \underset{(1,4)}{\begin{pmatrix} 1 & 2 & 3 & 4 \\ - & 1 & 2 & 3 \end{pmatrix}}, \underset{(4,3)}{\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & - \end{pmatrix}}, \underset{(3,2)}{\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & - & 4 \end{pmatrix}}, \underset{(2,1)}{\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & - & 3 & 4 \end{pmatrix}} \right\}$$

forms a generating set for $POI(4)$ since the edges of the graph Ω_X form a 4-cycle. By composing these edges, we can then generate all 16 edges of $POI_3(4)$ to get:

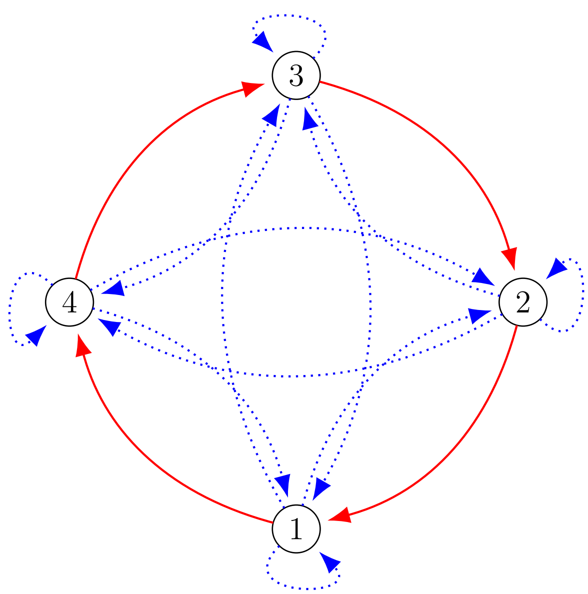


Figure 3: A graphical example of a generating set of $POI(4)$

Here, the solid directed arrows belong to the generating set, while the dotted directed arrows represent the remaining elements of $POI_3(4)$.

Example 3.6 The set X given by

$$\left\{ \underset{(5,3)}{\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 4 & 5 & - \end{pmatrix}}, \underset{(4,5)}{\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & - & 4 \end{pmatrix}}, \underset{(3,1)}{\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & - & 4 & 5 \end{pmatrix}}, \underset{(2,4)}{\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & - & 2 & 3 & 5 \end{pmatrix}}, \underset{(1,2)}{\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ - & 1 & 3 & 4 & 5 \end{pmatrix}} \right\}$$

forms a generating set for $POI(5)$ since the edges of the graph Ω_X form a 5-cycle. From this 5-cycle, we can generate the remaining 20 edges that correspond to the remaining 20 elements of $POI_4(5)$:

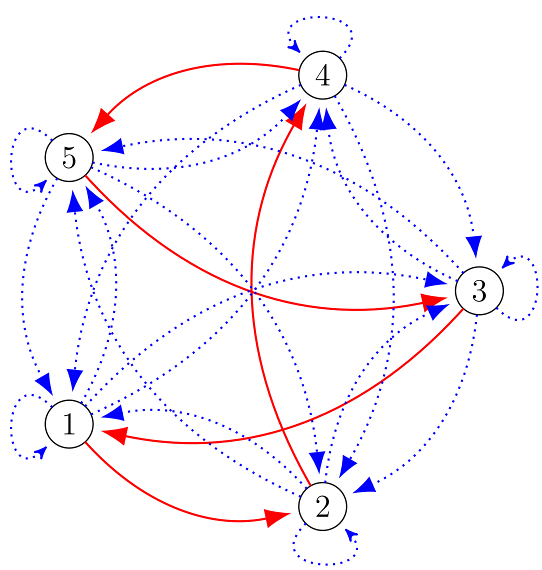


Figure 4: A graphical representation of a generating set of $POI(5)$

As in Example 3.5, the solid directed arrows belong to the generating set, while the dotted directed arrows represent the remaining elements of $POI_4(5)$.

Example 3.7 The set X given by

$$\left\{ \underset{(5,3)}{\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 4 & 5 & - \end{pmatrix}}, \underset{(3,5)}{\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & - & 3 & 4 \end{pmatrix}}, \underset{(4,1)}{\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & - & 5 \end{pmatrix}}, \underset{(1,2)}{\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ - & 1 & 3 & 4 & 5 \end{pmatrix}}, \underset{(2,4)}{\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & - & 2 & 3 & 5 \end{pmatrix}} \right\}$$

is not a generating set for $POI(5)$ since Ω_X forms two subcycles of the 5-cycle. By composing the edges in these subcycles we only generate 13 out of the 25 edges of $POI_4(5)$, as shown in Figure 5.

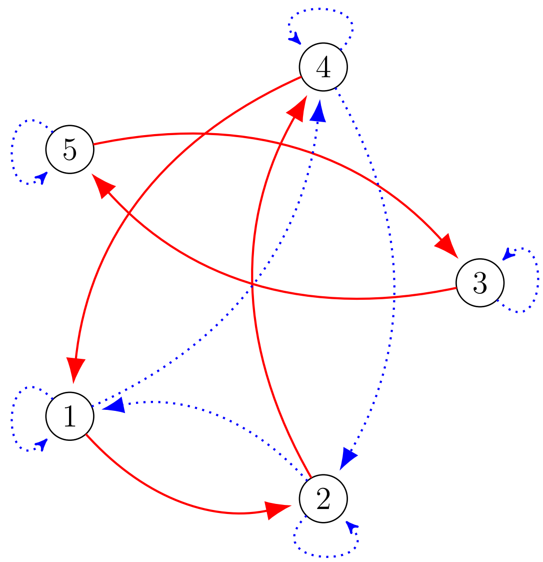


Figure 5: A graphical representation of a non-generating set of $POI(5)$

Theorem 3.8 *POI(n) has* $(n-1)!$ *minimal generating sets.*

Proof. From Theorem 3.3, we see that the number of minimal generating sets of $POI(n)$ is equal to the number of directed n-cycles that can be put on n vertices. It is a well-known observation in combinatorics that $(n-1)!$ such directed n-cycles can be constructed. $\square$

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