Discrete Approximation of Coloring the Plane*

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Abstract - This work is inspired by the open problem The Chromatic Number of The Plane. We approximate the plane by only considering points $(p/n, q/n) \in \mathbb{R}^2$ where p, q are any integers and a fixed positive integer n. To prevent triviality, we add the restriction that two points are *adjacent* if and only if their distance is between $1-\varepsilon$ and $1+\varepsilon$ for a non-negative ε. Our goal is to find the smallest ε for each n that will force us to use at least five colors. For $n = 2$, we show that the minimal ε is $1 - \frac{\sqrt{2}}{2}$ and prove this is also an upper bound for ε for all even $n \geq 4$. For $n = 3$, we prove a lower bound of $\frac{\sqrt{10}}{3} - 1$ and an upper bound of $1 - \frac{\sqrt{5}}{3}$. For odd $n \geq 11$, we prove that $1 - \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2n}$ is an upper bound.

Keywords : graph coloring; coloring the plane

Mathematics Subject Classification (2010) **:** 03G10; 06B20; 05C15; 05C63

1 Introduction

The Chromatic Number of The Plane, a problem in which the plane is made into a graph by treating points as vertices. Vertices are adjacent (i.e. they are connected by an edge) if and only if they are distance one apart. Each vertex in the plane is then assigned a color. The goal is then to find the minimal number of colors so that no two adjacent vertices have the same color. The chromatic number of the plane is not known, however past research has shown it is bounded between 5 and 7 [2], [3], [1].

Instead of working directly on the entire xy-plane, we are using a discrete approximation that consists only of the points $(p/n, q/n)$ where p, q are any integers and n is a fixed positive integer. We call this set of points "the lattice". We also fix ε to be a non-negative number. We color every point in the lattice, with the requirement that two points must have different colors if their distance is between $1-\varepsilon$ and $1+\varepsilon$ inclusively. If we take a lattice and create edges joining any two points whose distance falls between $1-\varepsilon$ and $1+\varepsilon$, we can partition those edges (see Figure 1). An edge class is defined by a distance d: if two points P and Q are distance d apart then they are in the edge class of distance d. For each n we are trying to find the smallest epsilon, $\varepsilon_{\min}(n)$, that requires at least five colors for a proper coloring. If no such ε exists, then we will define $\varepsilon_{\min}(n)$ as

*Partially supported by NSF grant DMS-1247679, Project PUMP: Preparing Undergraduates through Mentoring towards PhDs.

†Partially supported by the National Science Foundation grant DMS-1600778 and NASA MIRO NX15AQ06A.

the infimum among the set of all ε that require at least five colors for a proper coloring. Note that such an infimum always exists.

Obviously we need to test many ε to find $\varepsilon_{\min}(n)$. A good place to start is the classic checkerboard pattern, as this pattern always works when $\varepsilon = 0$. Next we increase the value of ε until the checkerboard pattern no longer works. The value of ε where this pattern is no longer a proper coloring is a lower bound for $\varepsilon_{\min}$. Alternatively, we could start with a very large value for ε. When ε is large there are lots of edges and edge classes, this makes it easy to find a K_5 or some subgraph that requires at least five colors to be properly colored. We then decrease the value of ε, which would decrease the number or edge classes and make it harder to find such a subgraph.

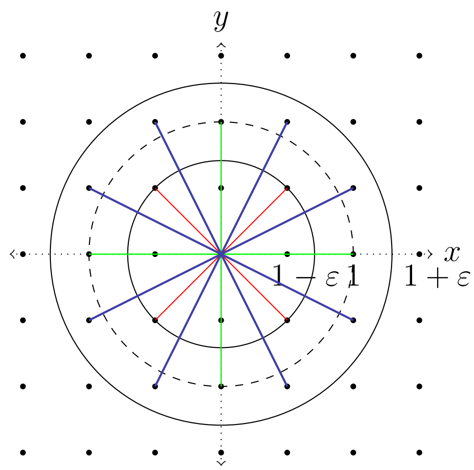

Figure 1: An annulus centered at $(0, 0)$ in $n = 2$ lattice when $\varepsilon = 1 - \frac{\sqrt{2}}{2}$ that highlights in different colors the edge classes incident to the point $(0, 0)$. Note that the thick blue edges connect to points that are members of the edge class of distance $\frac{\sqrt{5}}{2}$.

2 The Even n Cases

2.1 The $n = 2$ Case

We start with $n = 2$, as $n = 1$ is trivial.

Lower Bound

Theorem 2.1 $\varepsilon_{min}(2) \geq 1 - \frac{\sqrt{2}}{2}$.

Proof. When $\varepsilon = 0$ the lattice can be colored using 2 colors in a checkerboard pattern. We increase the value of ε until a new edge class is formed. When $\varepsilon = \frac{\sqrt{5}}{2} - 1$ the edge class of distance $\frac{\sqrt{5}}{2}$ is formed. For $\varepsilon = \frac{\sqrt{5}}{2} - 1$, there exists a repeating pattern of points such that four colors can be used to color the whole plane. Let $(0, 0)$, $(\frac{1}{2}, 0)$, $(0, \frac{1}{2})$ and $(\frac{1}{2}, \frac{1}{2})$ have color 1; let $(1, 0)$, $(0, \frac{3}{2})$, $(1, \frac{1}{2})$, and $(\frac{3}{2}, \frac{1}{2})$ have color 2; let $(0, 1)$, $(\frac{1}{2}, 1)$, $(0, \frac{3}{2})$, and $(\frac{1}{2}, \frac{3}{2})$ have color 3; and finally let $(1, 1)$, $(\frac{3}{2}, 1)$, $(1, \frac{3}{2})$, and $(\frac{3}{2}, \frac{3}{2})$ have color 4. These sixteen points form a square that is repeated in a grid pattern infinitely, and colors the whole plane. Thus when $\varepsilon = \frac{\sqrt{5}}{2} - 1$, the lattice can be filled with 4 colors in a checkerboard fashion as in Figure 2.

We then increase the value of ε until a new edge class is formed. The new edge class is formed when $\varepsilon = 1 - \frac{\sqrt{2}}{2}$ which creates the edge class of distance $\frac{\sqrt{2}}{2}$. When ε is at least $1 - \frac{\sqrt{2}}{2}$, two points that are directly diagonal, or points $\frac{\sqrt{2}}{2}$ units away from each other must have different colors so our checkerboard pattern no longer works. Therefore, $\varepsilon_{\min}(2) \geq 1 - \frac{\sqrt{2}}{2}$.

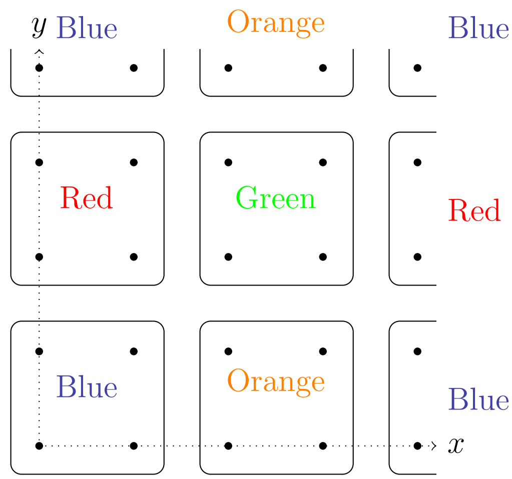

Figure 2: A four-coloring of the $n = 2$ lattice when ε is smaller than $1 - \frac{\sqrt{2}}{2}$.

$\square$

Upper Bound

Theorem 2.2 *Let n be a positive even integer. Then $\varepsilon_{min}(n) \leq 1 - \frac{\sqrt{2}}{2}$.*

Proof. Let $\varepsilon = 1 - \frac{\sqrt{2}}{2}$. Let A be the point $(0, \frac{1}{2})$, B be $(\frac{1}{2}, 1)$, C be $(1, \frac{1}{2})$, and D be $(\frac{1}{2}, 0)$. The distance between any pair of these points is between $1 - \varepsilon$ and $1 + \varepsilon$ inclusively. So these points make a K_4 and must all have different colors. Without loss of generality we can assign point A color 1, point B color 2, point C color 3, and point D color 4. Then, let E be $(0, 0)$. The point E can have either color 1 or color 4 since it is not adjacent to A and D (which have colors 1 and 4). This gives us 2 cases: when E is color 1 and when E is color 4. This subgraph is shown in Figure 3.

First, let's assume that E has color 1. Then, let F be $(0, 1)$. Point F must have color 2 because it must have a different color from E, C, and D. Then, let G be $(1, 1)$. Point G has color 3 since it must have a different color than F, A, and D. Let H be $(1, 0)$. Then H has color 4 since it needs to be assigned a color different from A, B, G. Then finally let I be at $(\frac{1}{2}, \frac{1}{2})$. I must have a fifth color since it must be colored differently from E, H, G, and F. This subgraph is shown in Figure 4. A similar argument works when E has color 4.

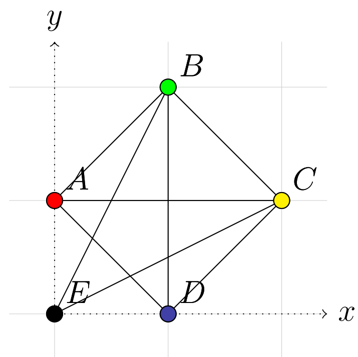

Figure 3: A subgraph that gives us 2 cases for finding the upper bound of $\varepsilon_{\min}(2)$.

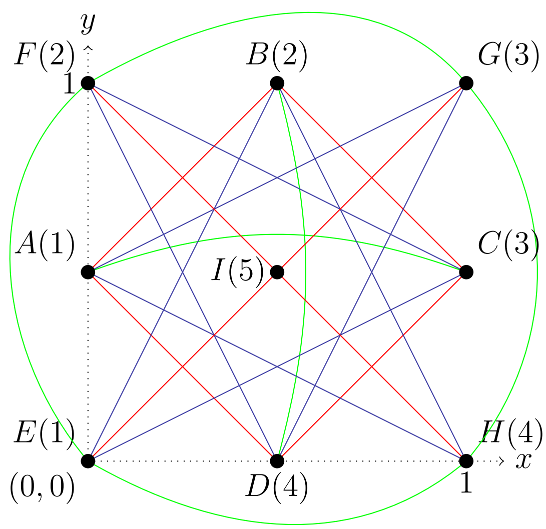

Figure 4: Case 1: E has color 1.

With these two cases, we know $\varepsilon_{\min}(n) \leq 1 - \frac{\sqrt{2}}{2}$. $\square$

Corollary 2.3 $\varepsilon_{min}(2) = 1 - \frac{\sqrt{2}}{2}$.

With these two theorems we have shown that $\varepsilon_{\min}(2) \leq 1 - \frac{\sqrt{2}}{2}$ and $\varepsilon_{\min}(2) \geq 1 - \frac{\sqrt{2}}{2}$.

3 The n = 3 Case

Lower Bound

To find a lower bound for $\varepsilon_{\min}(3)$, we need to find a coloring of the plane that uses only four colors. This coloring will have an associated ε that will become our lower bound.

Theorem 3.1 $\varepsilon_{\min}(3) > \frac{\sqrt{10}}{3} - 1$.

Proof. We start with C_1 (Figure 5), a coloring of the $n = 3$ lattice that uses only two colors, where all points distance 1 apart have different colors, and C_2 (Figure 6), a coloring of the $n = 3$ lattice which also uses only two colors, where all points distance $\frac{\sqrt{10}}{3}$ apart have different colors.

Let $C_1(P)$ be the color of the point P using the coloring C_1, and $C_2(P)$ be the color of the point P using C_2. We define a new coloring C (Figure 7) that uses four colors by combining C_1 and C_2 as follows: $C(P) = 1$ if $C_1(P) = 1$ and $C_2(P) = 1$; $C(P) = 2$ if $C_1(P) = 1$ and $C_2(P) = 2$; $C(P) = 3$ if $C_1(P) = 2$ and $C_2(P) = 1$; $C(P) = 4$ if $C_1(P) = 2$ and $C_2(P) = 2$.

It is clear that if $C(P) = C(Q)$, then $C_1(P) = C_1(Q)$ and $C_2(P) = C_2(Q)$, and consequently C is a coloring where all points that are either 1 or $\frac{\sqrt{10}}{3}$ apart are colored differently

Since there are no other distances present in this lattice such that

$$\frac{\sqrt{10}}{3} \geq d(P, Q) \geq 1 \geq 2 - \frac{\sqrt{10}}{3},$$

we let $\varepsilon = \frac{\sqrt{10}}{3} - 1$. So, when $\varepsilon = \frac{\sqrt{10}}{3} - 1$, five colors are not needed to color the plane and $\varepsilon_{\min}(3) > \frac{\sqrt{10}}{3} - 1$. $\square$

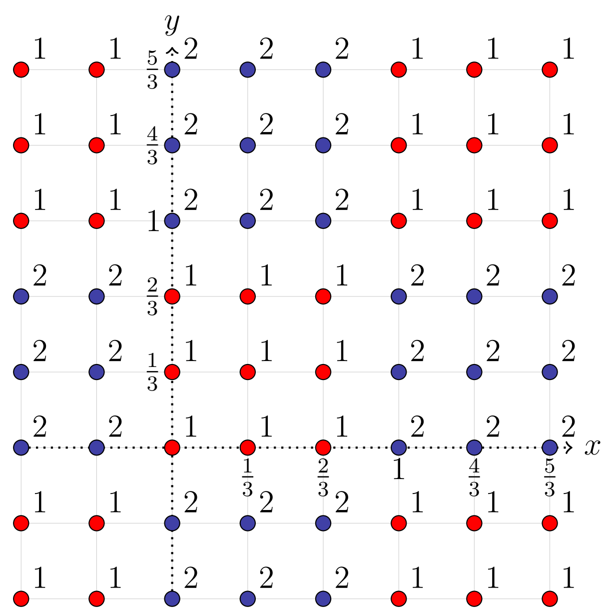

Figure 5: The coloring C_1 where all points distance 1 apart have different colors.

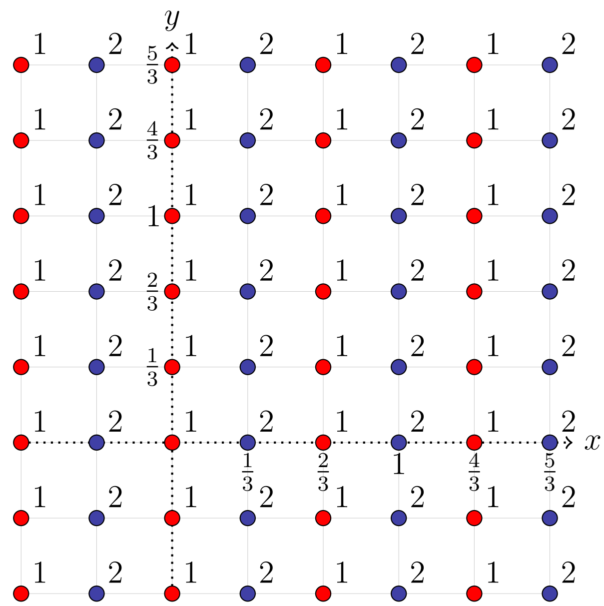

Figure 6: The coloring C_2 where all points distance $\frac{\sqrt{10}}{3}$ apart have different colors.

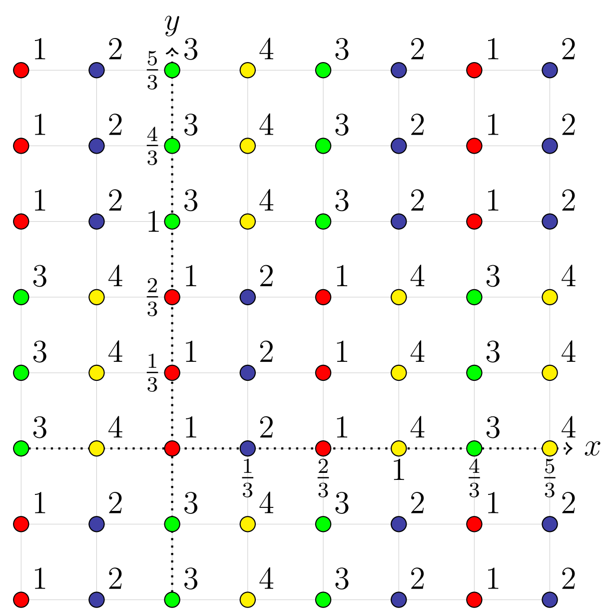

Figure 7: The coloring C.

Upper Bound

Theorem 3.2 $\varepsilon_{min}(3) \leq 1 - \frac{\sqrt{5}}{3}$.

Proof. Let $\varepsilon = 1 - \frac{\sqrt{5}}{3}$. Then in this case two points P and Q must have different colors when $\frac{\sqrt{5}}{3} \leq d(P,Q) \leq 2 - \frac{\sqrt{5}}{3}$.

Let A be the point $(0, 0)$, let B be the point $(0, 1)$, and let C be the point $(1, \frac{2}{3})$, see Figure 8. The only distances present between these three points are 1, $\frac{\sqrt{10}}{3}$, and $\frac{\sqrt{13}}{3}$. So, A, B, and C must be different colors and without loss of generality we can assign A color 1, B color 2, and C color 3. Then, let point D be $(\frac{1}{3}, \frac{1}{3})$. Then D must have either color 1 or 4. This gives us 2 cases.

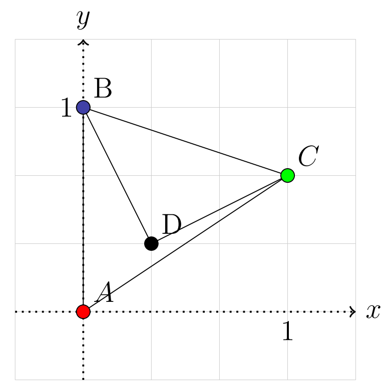

Figure 8: The formation of points that gives us 2 cases for the upper bound when $n = 3$.

Suppose D has color 1. Then let E be point $(\frac{2}{3}, \frac{4}{3})$, point E must have color 4 since it must be a different color than points B, D, and C. Also, let F be at point $(\frac{2}{3}, \frac{1}{3})$. F must have color 3 since it must be a different color than A, B, and E. Let point $(\frac{4}{3}, \frac{2}{3})$ be called G. Then G has color 2 since it must be colored differently than F, E, and D. Let point $(\frac{1}{3}, \frac{4}{3})$ be called H. H must have color 4 since it must have a different color than C, D, and G. After we determine all these points, a final point I at $(1, 1)$ must have color 5 since it cannot be the same color as B, D, F, and H. So, when D is color 1, at least five colors are needed to color this plane.

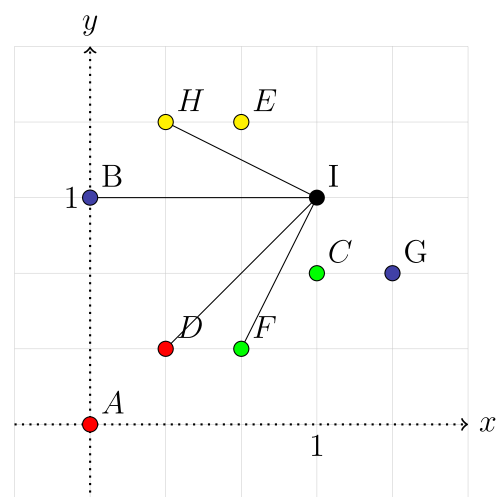

Figure 9: The formation of points when $n = 3$ and when D has color 1.

Suppose D is color 4. Then, let J be point $(\frac{2}{3}, -\frac{1}{3})$. J must have color 2 since it has to be a different color than A, C, and D. Also let K be at point $(\frac{4}{3}, 0)$. Then K must have color 1 since it must be colored differently than C, D, and J. Then let point $(\frac{1}{3}, 0)$ be called L. L must have color 4 because it has to be a different color than K, C, and B. Let M be at $(0, \frac{2}{3})$. Then M must have color 1 since it has to be a different color than points C, J, and L. After we determine all these points, a final point I at $(\frac{2}{3}, \frac{4}{3})$ must have color 5 since it cannot be the same color as B, C, D, and M. So, when D is color 4, at least five colors are needed to color this plane. □

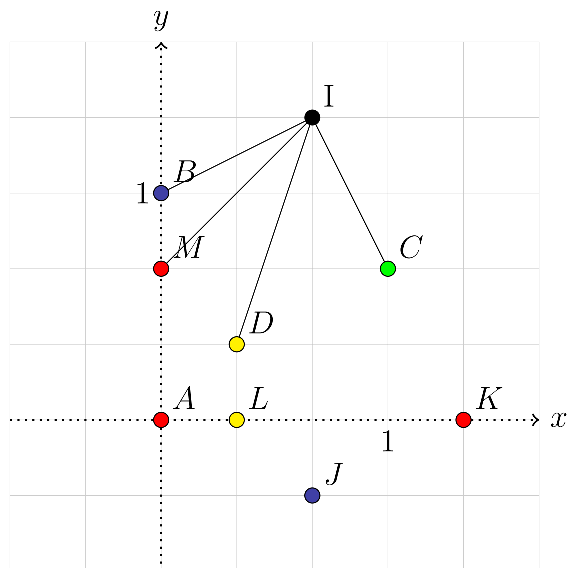

Figure 10: The formation of points when $n = 3$ and when D has color 4.

4 The Odd $n \geq 11$ Case

Theorem 4.1

$$\textit{If } n \geq 11 \textit{ is odd, then } \varepsilon_{min}(n) \leq 1 - \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2n}.$$

Proof. We can take the same subgraph that is used to prove $1 - \frac{\sqrt{2}}{2}$ is an upper bound for $\varepsilon_{min}(2)$; let's call it S. The graph S' will be a generalized S. The S' can be used to prove $1 - \frac{\sqrt{2}}{2}$ is an upper bound $\varepsilon_{min}(n)$ for greater values of n.

Let $A \in V(S)$ be the point $(0, \frac{1}{2})$. In S' the point A' will correspond to A. Let A' be the point $(0, \frac{n+1}{2n})$. Similarly, the point $F \in V(S)$ is $(0, 1)$, the point $(0, 1)$ is also in S' but we will define F' to be the point $(0, \frac{n+1}{2n})$ so that the distance between $(0, 0)$ and A' stays the same as the distance between A' and F'. We generalize the rest of the points in the same way. By doing this S' will consist of only three edge classes when $\varepsilon = 1 - \frac{\sqrt{2}}{2}$ just like S (see Figure 11).

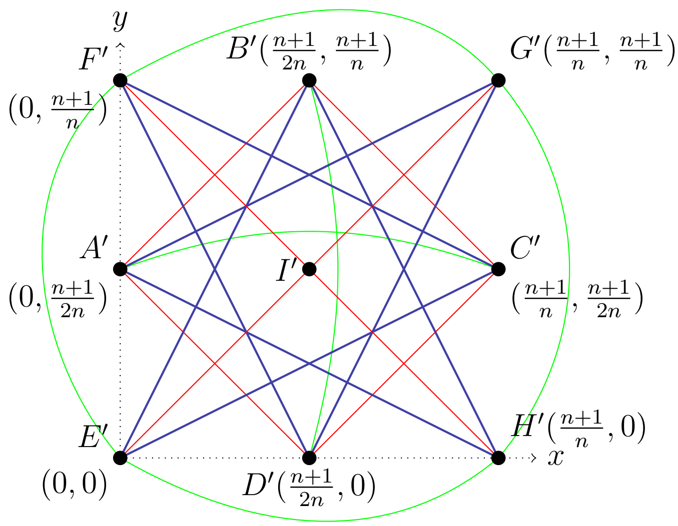

Figure 11: The generalized subgraph S'.

Using S', we prove that for all odd $n \geq 11$, $\varepsilon_{\min}(n) \leq 1 - \frac{\sqrt{2}}{2}$ is an upper bound. This argument is only valid for cases when n is odd and $(0, \frac{n+1}{2n})$ is in the n-lattice. Finally, we can improve the upper bound by measuring the distances of S'. By construction S' has only three edge classes, they are the edge classes of distances: $\frac{1+n}{n}$, $\frac{\sqrt{2}(1+n)}{2n}$, and $\frac{\sqrt{5}(1+n)}{2n}$. Each distance has an associated ε that is the smallest possible ε to create the edge class of said distance. The ε are: $\frac{1}{n}$, $\frac{\sqrt{5}}{2} + \frac{\sqrt{5}}{2n} - 1$, and $1 - \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2n}$. To see which ε will make the best upper bound, we graph the values and see which is the biggest, see Figure 12. When $n \leq 10$, $\varepsilon_{\min}(n) \leq \frac{\sqrt{5}}{2} + \frac{\sqrt{5}}{2n} - 1$. For $n \geq 11$, $1 - \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2n}$ is the biggest ε. If we round up to 11, we get that for odd $n \geq 11$, $\varepsilon_{\min}(n) \leq 1 - \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2n}$.

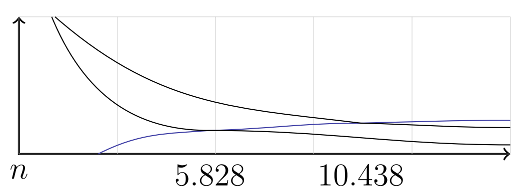

Figure 12: The graphs of the three ε: $\frac{1}{n}$, $\frac{\sqrt{5}}{2} + \frac{\sqrt{5}}{2n} - 1$, and $1 - \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2n}$.

□

Acknowledgments

We would like to thank Dr. Michael Krebs, our Faculty Advisor, at California State University, Los Angeles for providing valuable advice, for guiding us through these last few months and for his patience. We really appreciate the time, knowledge, and passion for mathematics he shared with us. This research project would not have been possible without him. We also thank the referees for their helpful comments.

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Received: October 6, 2017 **Accepted**: February 8, 2019
Communicated by Daphne Der-Fen Liu