Commuting Graphs of Split Metacyclic Groups

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Abstract - We study the link between groups and graphs created by considering the commuting graph of a group. We focus our efforts on groups that can be represented as the semidirect product of cyclic groups, and we describe the commuting graphs of two classes of such groups.

Keywords : commuting graph of a group; semidirect product; split metacyclic group

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This note is organized as follows. In Section 1 we recall a few definitions. In Section 2 we work with dihedral-like groups, and in Section 3 we approach groups with more complex structure.

1 Background

Given a group G, the center of G is

$$Z(G) = \{g \in G;\ gx = xg,\ \text{for all } x \in G\}.$$

Clearly, G is abelian if and only if $Z(G) = G$. When G is not abelian, we define the commuting graph of G, denoted $\mathcal{C}(G)$ by having vertex-set $G \setminus Z(G)$ and edges connecting vertices g_1 and g_2 if and only if $g_1 g_2 = g_2 g_1$.

In this note we obtain a simple presentation of $\mathcal{C}(G)$ in the case G is a semidirect product of certain cyclic groups. In this way, we generalize results obtained in [1], [5], [6], and [7].

A well-known object in group theory is the semidirect product of two groups (see, e.g., [2]). Since we will work with groups of this type, we define them next. Let H and K be groups and let $\phi : K \to Aut(H)$ be a homomorphism. In order to avoid confusion, we will use the notation $\phi(k) = \phi_k$. Let $G = H \times K$ be endowed with the operation:

$$(h, k)(a, b) = (h\phi_k(a), kb)$$

This multiplication makes G into a group, called the semidirect product of H and K, with respect to ϕ. We will denote this group by $H \rtimes_\phi K$. It is known that if a group

G has two subgroups, H and K, so that $HK = G$, $H \trianglelefteq G$, and $H \cap K = \{e\}$, then $G \cong H \rtimes_\phi K$, for some ϕ.

We recall now the definition of split metacyclic groups, which are semidirect products of cyclic groups:

Definition 1.1 *[3] A group is called* split metacyclic *if it has the following presentation:*

$$G_{\alpha,\beta,\gamma} =< a, b; a^\alpha = b^\beta = 1, \;\; aba^{-1} = b^\gamma >,$$

where $\alpha, \beta, \gamma \in \mathbb{N}$, and $\beta | \gamma^\alpha - 1$ (note that this implies that $\gcd(\beta, \gamma) = 1$*).*

We remark that these groups are called metacyclic on page 462 of [4]. As remarked in [3], the integers α, β, γ do not identify the isomorphism type of the group. The example given in [3] is $G_{6,36,19} \simeq G_{18,12,7}$, but a simpler example is $G_{3,7,2} \simeq G_{3,7,4}$ (see the discussion at the beginning of Section 3).

We have that $G_{\alpha,\beta,\gamma} = \mathbb{Z}_\beta \rtimes_\phi \mathbb{Z}_\alpha$, where $\phi \in Aut(\mathbb{Z}_\beta)$ is defined by $\phi(b) = b^\gamma$ (we write the operation of $\mathbb{Z}_\beta$ as multiplication instead of addition).

2 Dihedral-like Groups

We start by considering dihedral-like groups, which are split metacyclic groups $G^i_{2n} = G_{n,2,i}$ with $i > 1$. By Definition 1.1, it follows that they can be presented as

$$G^i_{2n} =< s, r; \; r^n = s^2 = e, \; srs^{-1} = r^i >$$

where $n, i \in \mathbb{N}$, $n > 1$, and $1 < i < n$ ($i = 1$ is uninteresting to us in this paper, as $G^1_{2n} \simeq \mathbb{Z}_n \times \mathbb{Z}_2$ is abelian) satisfies $i^2 \equiv 1 (\text{mod } n)$. In conclusion, we can simply say that i has order 2 modulo n. Note that $G^i_{2n} \simeq \mathbb{Z}_n \rtimes \mathbb{Z}_2$, for every i, and that $G^{n-1}_{2n} \simeq D_{2n}$ (the standard dihedral group of order $2n$).

Lemma 2.1 *Let $n, i \in \mathbb{N}$, $1 < i < n$, and $d = \gcd(i-1, n)$. Then, $Z(G^i_{2n}) =< r^{n/d} >$, and thus $|Z(G^i_{2n})| = d$.*

Proof. Assume that $sr^k \in Z(G^i_{2n})$. This element would commute with every element of the form r^j. Hence, we get

$$sr^{k+j} = (sr^k)(r^j) = (r^j)(sr^k) = s(sr^js)r^k = sr^{ij}r^k = sr^{ij+k}$$

It follows that $k + j \equiv ij + k \pmod n$, and thus $j \equiv ij \pmod n$, for every j. This is false, and so $sr^k \notin Z(G^i_{2n})$, for all k.

Now we assume that $r^k \in Z(G^i_{2n})$. This element commutes with, at least, all elements of the form r^j. We only need to check when it would commute with s. We get

$$sr^k = r^k \; s = s(sr^ks) = sr^{ik}$$

and thus $k \equiv ik \pmod n$, which implies $n \mid k(i-1)$. It follows that $\frac{n}{d}$ divides $k \cdot \frac{i-1}{d}$, but this implies that $\frac{n}{d}$ divides k. Hence, k has the form $\frac{n}{d}t$, for $t = 1, 2, \ldots, d$. □

Theorem 2.2 *Let $n, i \in \mathbb{N}$, $1 < i < n$, and $d = \gcd(i-1, n)$. Then, $\mathcal{C}(G_{2n}^i)$ is the disjoint union of $d+1$ complete graphs; one K_{n-d} and $\frac{n}{d}$ copies of K_d.*

Proof. The vertex-set of $\mathcal{C}(G_{2n}^i)$ contains all the elements of the form sr^k, and the elements r^k, for $k \neq \frac{n}{d}t$, for $t = 1, 2, \ldots, d$.

We know that elements in $< r >$ commute with each other. Now, none of the elements in $< r > \setminus < r^{n/d} >$ commutes with elements of the form sr^k, as if any did then they would commute with s, and thus would be in the center. Hence, elements in $< r > \setminus < r^{n/d} >$ commute only among themselves. This yields a complete graph on $n - d$ vertices in $\mathcal{C}(G_{2n}^i)$.

Next we check when $(sr^k)(sr^j) = (sr^j)(sr^k)$ occurs. Assuming this, we get:

$$r^{ik+j} = r^{ik}r^j = (sr^k s)r^j = (sr^k)(sr^j) = (sr^j)(sr^k) = (sr^j s)r^k = r^{ij}r^k = r^{ij+k}$$

which implies $ik + j \equiv ij + k \pmod{n}$. We re-write this equation as $i(k-j) \equiv k - j \pmod{n}$, and notice that this equation was already solved in the proof of Lemma 2.1. It follows that $k - j$ has the form $\frac{n}{d}t$, for $t = 1, 2, \ldots, d$. This yields $\frac{n}{d}$ complete graphs on d vertices in $\mathcal{C}(G_{2n}^i)$. □

Remark 2.3 Since we know that the structure of $\mathcal{C}(G_{2n}^i)$ is a disjoint union of complete graphs, it is now easy to find standard values associated to graphs, such as the minimum/maximum degree, diameter, chromatic number, etc. These parameters were part of the motivation given by authors in [1], [5], [6], and [7].

3 Groups of Order nq, for q Prime and $n \mid q - 1$

We start by recalling a well-known construction. Consider a non-abelian group G_{pq}, of order pq, where p and q are odd primes and $p < q$. Since G_{pq} is non-abelian, we must have that $p \mid q - 1$, $Z(G_{pq}) = \{e\}$, and $G_{pq} \cong \mathbb{Z}_q \rtimes \mathbb{Z}_p$. Moreover, the structure of G_{pq} does not depend on ϕ (see [2], Section 5.5), and so we can present it as follows:

$$G_{pq} =< x, y;\ x^q = y^p = e,\ yxy^{-1} = x^z >$$

where z has order p in $\mathbb{Z}_q^*$. With the notation of Definition 1.1, we have that $G_{pq} = G_{p,q,z}$ for all $z \neq 1$.

Instead of considering this group, we will next look at the non-abelian split metacyclic group $G_{n,q,z}$, where q is an odd prime and z has order n modulo $q - 1$. We will denote such a group by G_{nq}^z. This is a non-abelian group, of order nq, isomorphic to $\mathbb{Z}_q \rtimes_\phi \mathbb{Z}_n$, for some homomorphism $\phi : \mathbb{Z}_n \to Aut(\mathbb{Z}_q)$, and has the following presentation

$$G_{nq}^z =< x, y;\ x^q = y^n = e,\ yxy^{-1} = x^z >$$

where $1 < z < q$ has order n modulo q. That is, $\gcd(z, q) = 1$, $z^n \equiv 1 \pmod{q}$, and $z^i \not\equiv 1 \pmod{q}$, for all $1 \leq i < n$. Note that this implies that $n|q - 1$.

Remark 3.1 In G_{nq}^z, we get

$$y^j x y^{-j} = y^{j-1}(yxy^{-1})y^{-j+1} = y^{j-1}(x^z)y^{-j+1} = (y^{j-1}xy^{-j+1})^z$$

and so, an induction argument yields

$$y^j x y^{-j} = x^{z^j}$$

for all $j \in \mathbb{N}$. Note that the expression above also works for $j = 0$.

Next, we find the general structure of $\mathcal{C}(G_{nq}^z)$.

Theorem 3.2 *Let $n, q, z \in \mathbb{N}$, where q is an odd prime, $n \mid q-1$, and $1 < z < q$ has order n modulo q. Then, $Z(G_{nq}^z) = \{e\}$ and $\mathcal{C}(G_{nq}^z)$ consists of $q+1$ disjoint graphs: one K_{q-1}, and q copies of K_{n-1}.*

Proof. Fix the element $x^a \in G_{nq}^z$, where $0 < a < q$. Clearly x^a commutes with all the elements in $< x >$. Now we will see what other elements commute with x^a. We take y^j, where $0 < j < n$, and assume x^a commutes with it. We get:

$$\begin{aligned} x^a y^j &= y^j x^a \\ x^a &= y^j x^a y^{-j} \\ x^a &= (y^j x y^{-j})^a \\ x^a &= x^{a \cdot z^j} \end{aligned}$$

which implies $a \equiv a \cdot z^j \pmod{q}$. Since q is prime and $0 < a < q$, we get that $z^j \equiv 1 \pmod{q}$. However, $0 < j < n$ and the order of z modulo q is n, a contradiction. It follows that x^a commutes only with the elements in $< x >$. Hence, $Z(G_{pq}^z) \cap < x >= \{e\}$, and thus that the degree of x^a in $\mathcal{C}(G_{nq}^z)$ is $q-2$ (we do not count e and x^a). Moreover, the vertex-set $< x > \backslash \{e\}$ induces a complete graph on $q-1$ vertices in $\mathcal{C}(G_{nq}^z)$.

Similarly, the vertex-set $< y > \backslash \{e\}$ induces a complete graph on $n-1$ vertices in $\mathcal{C}(G_{nq}^z)$. These two complete graphs are disjoint from all other vertices in $\mathcal{C}(G_{nq}^z)$.

Now fix the element $x^a y^b \in G_{nq}^z$, where $0 < a < q$ and $0 < b < n$. Assume it commutes with $x^i y^j$, where $0 < i < q$ and $0 < j < n$. One of the products yields:

$$\begin{aligned} (x^i y^j)(x^a y^b) &= x^i (y^j x^a y^{-j}) y^j y^b \\ &= x^i (y^j x y^{-j})^a y^{j+b} \\ &= x^i x^{a \cdot z^j} y^{j+b} \\ &= x^{i + a \cdot z^j} y^{j+b} \end{aligned}$$

Thus, assuming $(x^i y^j)(x^a y^b) = (x^a y^b)(x^i y^j)$ implies $x^{i+a\cdot z^j} y^{j+b} = x^{a+i\cdot z^b} y^{j+b}$. Hence, those two elements commute if and only if

$$i + a \cdot z^j \equiv a + i \cdot z^b \pmod{q}$$

which can be re-written as

$$a(z^j - 1) \equiv i(z^b - 1) \pmod q \tag{1}$$

We want to solve Equation (1) for i and j, under the assumptions of a, b, q, z are given, and that $0 < a, i < q$ and $0 < b, j < n$. However, instead of doing that, we will only count how many solutions we can find for a fixed pair a, b.

Note that none of the four factors in Equation (1) are congruent to zero modulo q, either by assumption or because the order of z modulo q is larger than both b and j. Hence, once $0 < j < n$ is fixed and using that q is prime, there is exactly one solution (modulo q) for i, namely

$$a(z^j - 1)(z^b - 1)^{-1} \equiv i \pmod q$$

where $(z^b - 1)^{-1}$ is the inverse of $(z^b - 1)$ modulo q.

It follows that $x^a y^b$ commutes with exactly $n - 1$ elements of G_{nq}, one of them being itself. Moreover, none of these elements is in the center of G_{nq} because each one of them commutes with only $n - 1$ elements. It follows that the degree of $x^a y^b$ in $\mathcal{C}(G_{nq})$ is $n - 2$.

Finally, we re-write

$$a(z^j - 1) \equiv i(z^b - 1) \pmod q$$

using that q is prime and that both $z^j - 1$ and $z^b - 1$ are not congruent to zero modulo q. We get

$$a(z^b - 1)^{-1} \equiv i(z^j - 1)^{-1} \pmod q$$

where inverses are taken modulo q. It follows that every two elements that commute with $x^a y^b$ must also commute with each other. Hence, the set of all elements commuting with $x^a y^b$ induce a complete graph in $\mathcal{C}(G_{nq})$. □

Just like we had for the dihedral-like groups, now that we completely know the structure of $\mathcal{C}(G_{nq})$, it would be easy for us to find important values usually associated to graphs, such as the minimum/maximum degree, diameter, chromatic number, etc.

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References

[1] F. Ali, M. Salman, S. Huang, On the commuting graph of dihedral group. *Comm. Algebra* **44** (2016), 2389–2401.

[2] D. Dummit, R. Foote, *Abstract algebra*, Third edition, John Wiley & Sons, Inc., Hoboken, NJ, 2004.

[3] S.P. Humphries, D.C. Skabelund, Character Tables of Metacyclic Groups, *Glasgow Math. J.*, **57** (2015), 387–400.

[4] S. Mac Lane, G. Birkhoff, *Algebra*, The Macmillan Co., New York; Collier-Macmillan Ltd., London 1967.

[5] Z. Raza, S. Faizi, Commuting graphs of dihedral type groups. *Appl. Math. E-Notes* **13** (2013), 221–227.

[6] T. Tamizh Chelvam, K. Selvakumar, S. Raja, Commuting graphs on dihedral group. *J. Math. and Comput. Sci.* **2** (2011), 402–406.

[7] J. Vahidi, A. Asghar Talebi, The commuting graphs on groups D_{2n} and Q_n. *J. Math. and Comput. Sci.* **1** (2010), 123–127.

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