An Elementary Proof of Weisman's Congruence When $p = 2$

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Abstract - We give an elementary approach to proving divisibility results for a class of binomial sums that are similar to Fleck's congruence. We use tools that are accessible to undergraduate students and in proving the divisibility results, we obtain additional interesting properties that we highlight in several parts of the paper.

Keywords : Fleck's congruence; mathematical induction; polynomial congruence; divisibility; jumps

Mathematics Subject Classification (2020) **:** 11A07

1 Introduction

Fleck's congruence considers certain binomial sums modulo a prime power. In 1913, Fleck discovered the following congruence [1] for a prime p and an integer q:

$$\sum_{k \equiv q \pmod{p}} (-1)^k \binom{n}{k} \equiv 0 \pmod{p^{\lfloor \frac{n-1}{p-1} \rfloor}},$$

where $n \geq p$ is an integer. Here, $\lfloor . \rfloor$ represents the floor function.

In 1977, Weisman [4] extended this result to obtain

$$\sum_{k \equiv q \pmod{p^\alpha}} (-1)^k \binom{n}{k} \equiv 0 \pmod{p^{\left\lfloor \frac{n-p^{\alpha-1}}{\phi(p^\alpha)} \right\rfloor}}, \tag{1}$$

where α, n are positive integers, $n \geq p^{\alpha-1}$, and ϕ is the Euler-phi function. Weisman used p-adic numbers to prove the result.

Since then, there have been many works that have related these congruences to different structures in mathematics and several different generalizations have been suggested. For some of the related work we refer to [2], [3] and [5].

In this work, we deduce Weisman's result for $p = 2$ using only elementary tools that are accessible to undergraduate students. The main tool that we use is mathematical induction. In proving the result we discover some additional properties as well. Namely,

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*This work was done as part of an undergraduate research class at NAU we prove the tightness of the results in Weisman's congruence for $p = 2$, meaning that $2^{\lfloor \frac{n-2^{e-1}}{2^{e-1}} \rfloor}$ is the highest power of 2 that divides all the binomial sums in (1). We also proved that a special combination of the binomial sums mentioned above have better divisibility properties.

The rest of the work is organized as follows. In Section 2 we set up some notation for the problem as well as some of the numerical data that we obtained in estimating the power of 2 that divides the coefficients. In section 3 we prove the main divisibility result. In section 4, we prove a tightness result. We then conclude with some connected material from coding theory and plans for future research.

2 Notations and our take on the problem

In what follows, we will be considering many variables. So, we would like to remark that all variables will be integers unless otherwise stated.

We will be considering Weisman's generalization of Fleck's congruence for $p = 2$. So, in our case the sums in 1 will take on the following form:

$$M_i(n,e) := \sum_{t \equiv i \pmod{2^e}} (-1)^t \binom{n}{t}. \tag{2}$$

We will be considering the powers of 2 that divide these coefficients, for $e \geq 1$. Note that for $e \geq 1$, and for any $i \in \mathbb{Z}$, if $k \equiv i \pmod{2^e}$, then k and i have the same parity. This means that if i is odd, then all the summands in $M_i(n,e)$ will be negative and they will all be positive for even i. Thus we will introduce:

$$N_i(n,e) := \sum_{t \equiv i \pmod{2^e}} \binom{n}{t}. \tag{3}$$

The above observation implies that $M_i(n,e) = \pm N_i(n,e)$ for all the possible cases, which also implies that $2^s | M_i(n,e)$ if and only if $2^s | N_i(n,e)$. In other words $\nu_2(M_i(n,e)) = \nu_2(N_i(n,e))$, where $\nu_2(x)$ denotes the highest power of 2 that divides 2.

So for us the problem will take on the following form:

Question: Given n and $e \geq 1$, find the *greatest* s such that $2^s | N_i(n,e)$ for all $i = 0, 1, \ldots, 2^e - 1$. This entails proving a divisibility property as well as a tightness result.

2.1 Polynomial Version and Experimental Results

The main observation that we make, which will help us construct the inductive steps can be given in the following lemma:

Lemma 2.1 *For positive integers n and e we have*

$$(1+x)^n \equiv N_0(n,e) + N_1(n,e)x + \cdots + N_{2^e-1}(n,e)x^{2^e-1} \pmod{x^{2^e} - 1}. \tag{4}$$

where the modulus is taken over $\mathbb{Z}[x]$.

Proof. By binomial expansion theorem, we have

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \cdots + \binom{n}{n}x^n.$$

Reducing modulo $x^{2^e} - 1$, we get the coefficient of x^i to be

$$\binom{n}{i} + \binom{n}{i+2^e} + \cdots = N_i(n,e).$$

$\square$

This observation helps us do experiments on finding the maximum power of 2 that divides the coefficients as well as see the *jumps*, i.e., the places where the power of 2 that divides $N(n,e)$ changes.

We also have a graphical image that visually shows the *jumps*, i.e., changes in the value of $f(n,e)$, for different values of n and e, as well as a table of values.

Definition 2.2 *For $n \geq 2^{e-1}$ and $e \geq 1$, we will denote by $f(n,e)$ the largest integer so that $2^{f(n,e)}|N_i(n,e)$ for all $i = 0, 1, \ldots, 2^e - 1$. In other words*

$$f(n,e) = min\{\nu_2(N_i(n,e)), \quad i = 0, 1, \ldots, 2^e - 1\}.$$

From 1, as well as our experimental results, we expect

$$f(n,e) = \left\lfloor \frac{n}{2^{e-1}} - 1 \right\rfloor = \left\lfloor \frac{n - 2^{e-1}}{2^{e-1}} \right\rfloor \tag{5}$$

if $n \geq 2^{e-1}$.

The proof will be divided into two parts consisting of sections 3 and 4. Section 3 will prove the divisibility, equivalent to saying that $f(n,e) \geq \left\lfloor \frac{n-2^{e-1}}{2^{e-1}} \right\rfloor$. Section 4 will be proving that this is tight, which will be equivalent to $f(n,e) = \left\lfloor \frac{n-2^{e-1}}{2^{e-1}} \right\rfloor$ for all n, e.

3 Divisibility

In this section, we will prove that $f(n,e) \geq \left\lfloor \frac{|n-2^{e-1}|}{2^{e-1}} \right\rfloor$ for all $e \geq 1$ and $n \geq 2^{e-1}$. We first prove the following lemma, which we will need later.

Lemma 3.1 $\binom{2^{e-1}}{j}$ *is an even integer, for $1 \leq j \leq 2^{e-1} - 1$.*

$n\backslash e$	1	2	3	4
1	0	0	0	0
2	1	0	0	0
3	2	0	0	0
4	3	1	0	0
5	4	1	0	0
6	5	2	0	0
7	6	2	0	0
8	7	3	1	0
9	8	3	1	0
10	9	4	1	0
11	10	4	1	0
12	11	5	2	0
13	12	5	2	0
14	13	6	2	0
15	14	6	2	0
16	15	7	3	1
17	16	7	3	1
18	17	8	3	1
19	18	8	3	1
20	19	9	4	1
21	20	9	4	1
22	21	10	4	1
23	22	10	4	1
24	23	11	5	2
25	24	11	5	2

$f(n,e)$, for $e \in \{1,2,3,4\}$, $n \in \{0,1,\ldots,25\}$.

Figure 1: Values for $f(n,e)$ which turn out to be $\left\lfloor \frac{n}{2^{e-1}} - 1 \right\rfloor$ if $n \geq 2^{e-1}$ and 0 if $n < 2^{e-1}$.

Proof. Notice that

$$\binom{2^{e-1}}{j} = \frac{(2^{e-1})(2^{e-1}-1)!}{j(j-1)!(2^{e-1}-j)!} = \frac{2^{e-1}}{j}\binom{2^{e-1}-1}{j-1},$$

which implies

$$j\binom{2^{e-1}}{j} = 2^{e-1}\binom{2^{e-1}-1}{j-1}.$$

We know 2^{e-2} is the highest power of 2 that can divide j, given the range considered

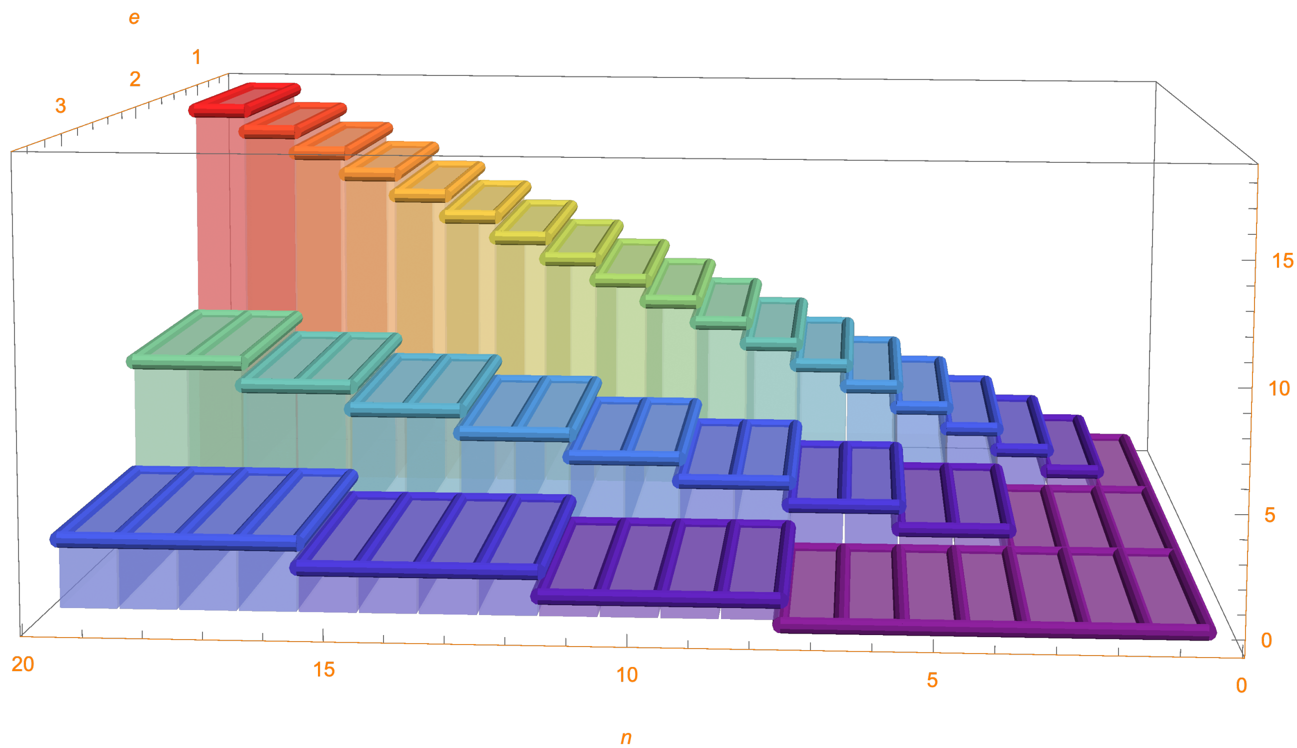

Figure 2: The *jumps* corresponding to different values of e and n. Here the values of e, n, and $f(n,e)$ are the x, y, and z axes respectively. Values of $f(n,e)$ for when $n, e = 0$ are omitted due to their trivial nature.

for j. So, the numerator of $\frac{2^{e-1}}{j}$ as a fraction in lowest terms is divisible by 2. Thus, $\binom{2^{e-1}}{j}$ is even. $\square$

We are now ready to prove the first main theorem, which will give us the divisibility result:

Theorem 3.2 *Suppose $n = k \cdot 2^{e-1}$, $k \geq 1$, and let*

$$(1+x)^{k \cdot 2^{e-1}} \equiv A_0 + A_1 x + \cdots + A_{2^e-1} x^{2^e-1} \pmod{x^{2^e} - 1}.$$

Then $2^{k-1} | A_i$ for all $i = 0, 1, \ldots 2^e - 1$ and $2^k | (A_i + A_{i+2^{e-1}})$, where the sum $i + 2^{e-1}$ is taken modulo 2^e, equivalently $f(k \cdot 2^{e-1}, e) \geq k - 1$.

Proof. We will prove this by inducting on k.
Let $k = 1$. Then, $2^0 | A_i$ is trivially true. Note that by Lemma 3.1, all the coefficients of $(1+x)^{2^{e-1}}$ are even except $A_0 = A_{2^{e-1}} = 1$, and so $A_0 + A_{2^{e-1}} = 2$, is even as well. So the base case checks out for both assertions.

We assume both assertions to be true for some $k \geq 1$ and we will prove them for $k+1$. We first observe that

$$(1+x)^{(k+1)2^{e-1}} = (1+x)^{2^{e-1}} (1+x)^{k \cdot 2^{e-1}},$$

which leads to the congruence

$$\sum_{i=0}^{2^e-1} B_i x^i \equiv (1+x)^{2^{e-1}} \left(\sum_{i=0}^{2^e-1} A_i x^i \right) \pmod{x^{2^e}-1},$$

where $B_i = N_i((k+1)2^{e-1}, e)$.

Expanding the binomial sum we get

$$\sum_{i=0}^{2^e-1} B_i x^i \equiv \left(\sum_{j=0}^{2^{e-1}} \binom{2^{e-1}}{j} x^j \right) (A_0 + A_1 x + \cdots + A_{2^e-1} x^{2^e-1}) \pmod{x^{2^e}-1}.$$

By collecting terms we can now express the coefficients of each B_i as a sum:

$$B_i = \sum_{j=0}^{2^{e-1}} \binom{2^{e-1}}{j} A_{i-j}. \tag{6}$$

In order to complete the induction, we must prove $2^k | B_i$ and $2^{k+1} | (B_i + B_{i+2^{e-1}})$ for all $0 \le i \le 2^e - 1$. For the first assertion, we rewrite (6) as

$$B_i = A_i + A_{i+2^{e-1}} + \sum_{j=1}^{2^{e-1}-1} \binom{2^{e-1}}{j} A_{i-j}. \tag{7}$$

By the induction hypothesis, $2^k | (A_i + A_{i+2^{e-1}})$ and $2^{k-1} | A_i$ for each $0 \le i \le 2^e - 1$. Moreover, by Lemma 3.1, $\binom{2^{e-1}}{j}$ is even for all $1 \le j \le 2^{e-1} - 1$, which implies $2^k | \sum_{j=1}^{2^{e-1}-1} \binom{2^{e-1}}{j} A_{i-j}$, hence leading to $2^k | B_i$ in (7).

For the second assertion, we observe that

$$\begin{aligned}
B_i + B_{i+2^{e-1}} &\equiv \sum_{j=0}^{2^{e-1}} \binom{2^{e-1}}{j} A_{i-j} + \sum_{j=0}^{2^{e-1}} \binom{2^{e-1}}{j} A_{i+2^{e-1}-j} \pmod{x^{2^e}-1} \\
&\equiv \sum_{j=0}^{2^{e-1}} \binom{2^{e-1}}{j} (A_{i-j} + A_{i+2^{e-1}-j}) \\
&\equiv (A_i + A_{i+2^{e-1}}) + (A_{i-2^{e-1}} + A_i) + \sum_{j=1}^{2^{e-1}-1} \binom{2^{e-1}}{j} (A_{i-j} + A_{i+2^{e-1}-j}).
\end{aligned}$$

Notice: $A_i + A_{i+2^{e-1}}$, and $A_{i-2^{e-1}} + A_i$ occur when $j = 0$, 2^{e-1} respectively. Because $i - 2^{e-1} \equiv i + 2^{e-1} \pmod{2^e}$ we can say

$$B_i + B_{i+2^{e-1}} \equiv 2(A_i + A_{i+2^{e-1}}) + \sum_{j=1}^{2^{e-1}-1} \binom{2^{e-1}}{j} (A_{i-j} + A_{i+2^{e-1}-j}). \tag{8}$$

Again, by the induction hypothesis, $2^k|(A_i + A_{i+2^{e-1}})$, therefore $2^{k+1}|2(A_i + A_{i+2^{e-1}})$. In the same way, $2^k|(A_{i-j} + A_{i+2^{e-1}-j})$ for $1 \leq j \leq 2^{e-1} - 1$, and by Lemma 3.1, we know $\binom{2^{e-1}}{j}$ is even for all j with $1 \leq j \leq 2^{e-1} - 1$, which implies that, $2^{k+1}|(B_i + B_{i+2^{e-1}})$ in (8). This completes the induction. □

The following is an immediate corollary of Theorem 3.2:

Corollary 3.3 *For $k \cdot 2^{e-1} \leq n < (k+1)2^{e-1}$, $f(n, e) \geq k - 1$.*

Proof. Considering the range for n we will let

$$(1 + x)^n = (1 + x)^{k2^{e-1}}(1 + x)^{n-k2^{e-1}}.$$

Then,

$$\begin{aligned}(1 + x)^n &= \left(\sum_{i=0}^{2^e-1} A_i x^i\right)\left(1 + \binom{n - k2^{e-1}}{1}x + \cdots + x^{n-k2^{e-1}}\right)\\ &\equiv \sum_{i=0}^{2^e-1} N_i(n, e)x^i \pmod{x^{2^e} - 1},\end{aligned}$$

which implies $N_i(n, e)$ can be written as a linear combination of the A_i coefficients. Since, the A_i coefficients are divisible by 2^{k-1}, from Theorem 3.2 we have that $2^{k-1}|N_i(n, e)$ for each $i = 0, 1, \ldots, 2^e - 1$. □

Corollary 3.4 *For all positive integers n and e, we have $f(n, e) \geq \left\lfloor \frac{|n-2^{e-1}|}{2^{e-1}} \right\rfloor$.*

Proof. Combining *Theorem 3.2.* and *Corollary 3.3* we have that $f(n, e) \geq k - 1$ for $k \cdot 2^{e-1} \leq n < (k + 1)2^{e-1}$, for all $k \geq 1$, which implies

$$f(n, e) \geq \left\lfloor \frac{|n - 2^{e-1}|}{2^{e-1}} \right\rfloor.$$

In particular, if $n \geq 2^{e-1}$, then

$$f(n, e) \geq \left\lfloor \frac{n - 2^{e-1}}{2^{e-1}} \right\rfloor.$$

□

4 Tightness

In this section we will prove that $f(n, e) = \left\lfloor \frac{|n-2^{e-1}|}{2^{e-1}} \right\rfloor$ for all the possible values of n and e. We will need a few preliminary results.

Lemma 4.1 *Let $e \geq 1$ be an integer. Then $\binom{2^e-1}{j}$ is odd for all j with $1 \leq j \leq 2^e - 2$.*

Proof. We start with

$$\binom{2^e-1}{j} = \frac{(2^e-1)!}{j!(2^e-1-j)!} = \frac{(2^e-1)(2^e-2)\dots(2^e-j)}{1\cdot 2\cdot \dots \cdot j},$$

which implies

$$\binom{2^e-1}{j} = \prod_{i=1}^{j} \frac{2^e-i}{i}.$$

Here we will let $i = 2^{r_i}m_i$, $1 \le i \le j$, with $\gcd(2, m_i) = 1$. Considering the range for j and the fact that $i \le j$ we can say that $r_i < e$ for all i. A general term in the product is of the form

$$\frac{2^e-i}{i} = \frac{2^e-2^{r_i}m_i}{2^{r_i}m_i} = \frac{2^{e-r_i}-m_i}{m_i},$$

which must be odd since $r_i < e$ and m_i is an odd integer. Since each term in the product is odd, we conclude that $\prod_{i=1}^{j}\frac{2^e-i}{i}$ is odd, which completes the proof. □

Lemma 4.2 *For any integer $e \ge 2$ we have $2||\binom{2^{e-1}}{2^{e-2}}$, i.e., $2|\binom{2^{e-1}}{2^{e-2}}$ but $2^2 \nmid \binom{2^{e-1}}{2^{e-2}}$.*

Proof. The proof is similar to the proof of Lemma 3.1. So we start with

$$\binom{2^{e-1}}{2^{e-2}} = \frac{(2^{e-1})(2^{e-1}-1)!}{(2^{e-2})(2^{e-2}-1)!(2^{e-2})!} = 2\binom{2^{e-1}-1}{2^{e-2}-1}.$$

But $\binom{2^{e-1}-1}{2^{e-2}-1}$ is odd by Lemma 4.1, which implies $2||\binom{2^{e-1}}{2^{e-2}}$. □

Lemma 4.3 *Let $e \ge 1$ be an integer. Then $4|\binom{2^{e-1}}{j}$ for all j with $1 \le j \le 2^{e-2}-1$.*

Proof. In a similar way as was done before we start with

$$\binom{2^{e-1}}{j} = \frac{(2^{e-1})(2^{e-1}-1)!}{j(j-1)!(2^{e-1}-j)!} = \frac{2^{e-1}}{j}\binom{2^{e-1}-1}{j-1}.$$

Considering the range for j, we have $j = 2^{r_j}m_j$ where $r_j \le e-3$, and $(2, m_j) = 1$. This implies

$$\frac{2^{e-1}}{j} = \frac{2^{e-1}}{2^{r_j}m_j} = \frac{2^{e-r_j-1}}{m_j}.$$

We have

$$m_j\binom{2^{e-1}}{j} = 2^{e-r_j-1}\binom{2^{e-1}-1}{j-1},$$

which implies $2^{e-r_j-1}|\binom{2^{e-1}}{j}$, since m_j is odd. But since $r_j \le e-3$, this implies $e-r_j-1 \ge 2$, which then implies

$$2^2 \mid \binom{2^{e-1}}{j}.$$

□

Lemma 4.4 *Let $A_i = N_i((k+1)2^{e-1}-1, e)$ for $k \geq 1$ and $i = 0, 1, \ldots, 2^e - 1$. Then $2^{k+1} | (A_i + A_{i+2^{e-1}})$, where the sum $i + 2^{e-1}$ is taken modulo 2^e.*

Proof. We will induct on k.

When $k = 1$, $A_i + A_{i+2^{e-1}} = \binom{2^e-1}{i} + \binom{2^e-1}{i+2^{e-1}}$, which is the coefficient of x^i when we reduce $(1+x)^{2^e-1}$ modulo $x^{2^{e-1}} - 1$. But by Theorem 3.2, we know that this is divisible by $\left\lfloor \frac{2^e-1-2^{e-2}}{2^{e-2}} \right\rfloor = \left\lfloor 3 - \frac{1}{2^{e-2}} \right\rfloor = 2$, so the base case checks out.

Assume the assertion to be true for some $k \geq 1$, and let us prove it for $k+1$.

We observe that $(1+x)^{(k+2)2^{e-1}-1} = (1+x)^{(k+1)2^{e-1}-1}(1+x)^{2^{e-1}}$, which with the same allocation of A_i's and B_i's as in the proof of Theorem 3.2, leads to

$$B_i + B_{i+2^{e-1}} \equiv 2(A_i + A_{i+2^{e-1}}) + \sum_{j=1}^{2^{e-1}-1} \binom{2^{e-1}}{j} (A_{i-j} + A_{i+2^{e-1}-j}).$$

Since, by induction hypothesis, $A_i + A_{i+2^{e-1}}$ and $A_{i-j} + A_{i+2^{e-1}-j}$ are all divisible by 2^{k+1} and by Lemma 3.1, $\binom{2^{e-1}}{j}$ is even, hence $2^{k+2} | (B_i + B_{i+2^{e-1}})$, completing the induction. $\square$

We are now ready to prove the following theorem, which is a special case of a more general result.

Theorem 4.5 $2^{k-1} || N_i((k+1)2^{e-1}-1, e)$ *for all $k \geq 1$ and $i = 0, 1, \ldots, 2^e - 1$.*

Proof. We will prove the result using induction on k.

If $k = 1$, then we have that $n = 2^e - 1$, and so

$$(1+x)^{2^e-1} = \sum_{i=0}^{2^e-1} \binom{2^e-1}{i} x^i,$$

which means $N_i(2^e - 1, e) = \binom{2^e-1}{i}$, which is odd by Lemma 4.1 for $i = 0, 1, \ldots, 2^e - 1$. Thus the base case holds.

Inductively assuming the assertion to be true for some $k \geq 1$, consider the $k+1$-th case. With $A_i = N_i((k+1)2^{e-1}-1, e)$, the induction hypothesis states that $2^{k-1} || A_i$, for all $i = 0, 1, \ldots, 2^e - 1$. Then, for $k+1$, we obtain the following coefficients:

$$\begin{aligned}
\sum_{i=0}^{2^e-1} B_i x^i &\equiv (1+x)^{(k+2)2^{e-1}-1} \\
&\equiv (1+x)^{(k+1)2^{e-1}-1}(1+x)^{2^{e-1}} \\
&\equiv (A_0 + A_1 x + \cdots + A_{2^e-1} x^{2^{e-1}})(1+x)^{2^{e-1}} \pmod{x^{2^e} - 1},
\end{aligned}$$

which leads to

$$
\begin{aligned}
B_i &= \sum_{j=0}^{2^{e-1}} \binom{2^{e-1}}{j} A_{i-j} \\
&= A_i + A_{i+2^{e-1}} + \sum_{j=1}^{2^{e-1}-1} \binom{2^{e-1}}{j} A_{i-j}. \\
&= A_i + A_{i+2^{e-1}} + \binom{2^{e-1}}{2^{e-2}} A_{i-2^{e-2}} + \sum_{j=1,\, j\neq 2^{e-2}}^{2^{e-1}-1} \binom{2^{e-1}}{j} A_{i-j}.
\end{aligned}
$$

Now, by Lemma 4.4, $A_i + A_{2^{e-1}}$ is divisible by 2^{k+1}. On the other hand, by our induction hypothesis (or by Theorem 3.2), each A_{i-j} in the sum is divisible by 2^{k-1} and by Lemma 4.3, the coefficients in the sums are all divisible by 4, which means, we have

$$
B_i \equiv \binom{2^{e-1}}{2^{e-2}} A_{i-2^{e-2}} \pmod{2^{k+1}}.
$$

However, by Lemma 4.2, $2||\binom{2^{e-1}}{2^{e-2}}$ and by induction hypothesis, $2^{k-1}||A_{i-2^{e-2}}$, which means $B_i \equiv 2^k \pmod{2^{k+1}}$ or that $2^k||B_i$, thus completing the induction step. $\square$

We are now ready to obtain the main tightness result as a corollary of Theorem 4.5:

Corollary 4.6 *$f(n,e) = \left\lfloor \frac{n-2^{e-1}}{2^{e-1}} \right\rfloor$ for all positive integers n and e with $n \geq 2^{e-1}$. In other words,*

$$
\min\{\nu_2(N_i(n,e)) |\, i = 0,1,2,\ldots,2^e-1\} = \left\lfloor \frac{n-2^{e-1}}{2^{e-1}} \right\rfloor.
$$

Proof. Suppose k is such that $k \cdot 2^{e-1} \leq n \leq (k+1)2^{e-1} - 1$. Then by Theorem 3.2, $2^{k-1}|N_i(n,e)$ for all $i = 0,1,\ldots,2^e-1$. We claim that
$k-1 = \min\{\nu_2(N_i(n,e)) |\, i = 0,1,2,\ldots,2^e-1\}$, for n in the stated range. Let us assume the converse so that $2^k|N_i(n,e)$ for all $i = 0,1,\ldots,2^e-1$. Then we have

$$
(1+x)^{(k+1)2^{e-1}-1} = (1+x)^n (1+x)^{(k+1)2^{e-1}-1-n},
$$

meaning that we can obtain $N_i((k+1)2^{e-1}-1,e)$ as linear combinations of the $N_i(n,e)$'s. This would imply that $N_i((k+1)2^{e-1}-1,e)$ would be divisible by 2^k for all i, contradicting Theorem 4.5. Thus we must have

$$
\min\{\nu_2(N_i(n,e)) |\, i = 0,1,2,\ldots,2^e-1\} = k-1 = \left\lfloor \frac{n-2^{e-1}}{2^{e-1}} \right\rfloor.
$$

$\square$

5 Conclusion

We have proved Weisnan's congruence in the case $p=2$, using an elementary approach. We also proved a tightness result. In doing so, we obtained several interesting results that were not found in the literature. In particular, we proved that

$$2^k \left| \left(\sum_{t\equiv i \pmod{2^e}} \binom{k2^{e-1}}{t} + \sum_{t\equiv 2^{e-1}+i \pmod{2^e}} \binom{k2^{e-1}}{t} \right) \right.,$$

$$2^{k+1} \left| \left(\sum_{t\equiv i \pmod{2^e}} \binom{(k+1)2^{e-1}-1}{t} + \sum_{t\equiv 2^{e-1}+i \pmod{2^e}} \binom{(k+1)2^{e-1}-1}{t} \right) \right.,$$

and

$$2^{k-1} \left\| \sum_{t\equiv i \pmod{2^e}} \binom{(k+1)2^{e-1}-1}{t} \right.,$$

$$i = 0,1,2,\ldots,2^e-1.$$

These Fleck-like congruences have many applications in the literature. One particular application can be found in [5], where a divisibility result was found for the Lee weight enumerators of codes over $\mathbb{Z}_4$. The results found above can be used in proving that the results found in [5] are best possible, because the Lee weight enumerator of the trivial code, namely $\mathbb{Z}_4^k$ is given by $(1+2x+x^2)^k$, which is equivalent to $(1+x)^{2k}$.

Acknowledgments

The authors would like to thank the anonymous referees for their valuable comments and suggestions that helped improve the paper profoundly.

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Received: November 15, 2019 **Accepted**: July 10, 2020
Communicated by Kristen Stagg Rovira