Canonical Expressions of Algebraic Curvature Tensors

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Abstract - Algebraic curvature tensors can be expressed in a variety of ways, and it is helpful to develop invariants that can distinguish between them. One potential invariant is the signature of R, which could be defined in a number of ways, similar to the signature of an inner product. This paper shows that any algebraic curvature tensor defined on a vector space V with $dim(V) = n$ can be expressed using only canonical algebraic curvature tensors from forms with rank k or higher for any $k \in \{2, \dots, n\}$, and that such an expression is not unique, eliminating some possibilities for what one might define the signature of R to be. We also provide bounds on the minimum number of algebraic curvature tensors of rank k needed to express any given R.

Keywords : canonical algebraic curvature tensor; signature conjecture; linear independence

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1 Introduction

Throughout, V is a real vector space with finite dimension n. A multilinear function $R : V \times V \times V \times V \to \mathbb{R}$ is an *algebraic curvature tensor* if $\forall x, y, z, w \in V$, R satisfies

$$R(x, y, z, w) = R(z, w, x, y) = -R(y, x, z, w), \text{ and}$$
$$R(x, y, z, w) + R(x, z, w, y) + R(x, w, y, z) = 0.$$

The space of all algebraic curvature tensors on V is denoted $\mathcal{A}(V)$. Given a symmetric bilinear form φ, we can define the *canonical algebraic curvature tensor* R_φ as

$$R_\varphi(x, y, z, w) = \varphi(x, w)\varphi(y, z) - \varphi(x, z)\varphi(y, w).$$

R_φ has the property that for any positive real number c, $R_{\sqrt{c}\varphi} = cR_\varphi$. Since algebraic curvature tensors are multilinear forms, R is determined by its values on some basis $\{e_i\}$. For brevity, $R(e_i, e_j, e_k, e_l)$ is denoted R_{ijkl}.

For every symmetric bilinear form φ, there is some basis $\{e_i\}$ where φ is diagonal. On this basis, the only potentially nonzero entries of R_φ are the $R_{\varphi_{ijji}}$ entries. Note that for any R, R_{jiij}, R_{ijij}, etc. are defined by their relation to a given R_{ijji} using the properties of algebraic curvature tensors. Thus it suffices to define R_φ by all the possible $R_{\varphi_{ijji}}$.

In [3], Gilkey showed that any algebraic curvature tensor R can be expressed in the form

$$R = \sum_{i=1}^{m} \epsilon_i R_{\varphi_i}$$

for $\epsilon_i = 1$ or -1 and some symmetric bilinear forms φ_i. For a given R, define

$$\nu(R) = \min\{m | R = \sum_{i=1}^{m} \epsilon_i R_{\varphi_i}\}.$$

For any positive integer n, define

$$\nu(n) = \max_{R \in \mathcal{A}(V)} \nu(R)$$

where V has dimension n.

For some positive integer $k \geq 2$, we define

$$\nu_k(R) = \min\{m | R = \sum_{i=1}^{m} \epsilon_i R_{\varphi_i}, \text{ where } \forall i, \text{ Rank}(\varphi_i) \geq k\}.$$

Then, for any positive integer n, we define

$$\nu_k(n) = \max_{R \in \mathcal{A}(V)} \nu_k(R)$$

where V has dimension n. Note that if $\text{Rank}(\varphi) = 1$ or 0, R_φ is the zero tensor[4]. Thus any minimal expression for $R \neq 0$ contains only forms of rank 2 or higher, so the absolute minimal number of canonical tensors needed, $\nu(R)$ is equal to $\nu_2(R)$ for all $R \neq 0$, and $\nu_2(n) = \nu(n)$. It was shown in [4] that $\nu(n) \leq \frac{n(n+1)}{2}$.

Any symmetric bilinear form φ can be diagonalized, and Sylvester's Law of Inertia [5] states that the number of negative entries p, the number of positive entries q, and the number of 0 entries s along the diagonal is unique. (p,q,s) is called the *signature* of φ.

Throughout, we denote diagonal matrices

$$\text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n) = \begin{bmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \lambda_n \end{bmatrix}.$$

Throughout the paper, we will often need to show that $R_\varphi = R_A + R_B$ for some nonzero symmetric bilinear forms φ, A, and B. We carefully demonstrate this the first time it arises in the proof of Theorem 2.1. All similar claims are proved in the same way, so we do not demonstrate the calculations again; rather, we describe any relevant differences in the constructions. For any symmetric bilinear form φ with Rank $(\varphi) \geq 3$, there is no ψ for which $R_\varphi = -R_\psi$ [1]. Noting this, the following conjecture was made.

Conjecture 1.1 (The Signature Conjecture) For any algebraic curvature tensor R and expression

$$R = \sum_{i=1}^{\nu_3(R)} \epsilon_i R_{\varphi_i}$$

where $\text{Rank}(\varphi_i) \geq 3 \ \forall i$, the number of i such that $\epsilon_i = -1$ is unique.

If one is presented with components of two algebraic curvature tensors on different bases that could perhaps be the same tensor, it is useful to develop quantities that can distinguish between these algebraic curvature tensors. These quantities are called *invariants*. If the signature conjecture were true, we could define the signature of an algebraic curvature tensor R to be the number of $+$ and $-$ signs used any expression of R in $\nu_3(R)$ terms, and the signature of R would be an invariant.

In Section 2, we show that $\nu_3(R)$ is well defined for every R. In Section 3, we show that the Signature Conjecture is not true as stated in Conjecture 1.1, and we provide revised conjectures in Section 4.

2 Bounds on $\nu_k(n)$

Gilkey's proof that $R = \sum_{i=1}^{m} \epsilon_i R_{\varphi_i}$ for every R requires that some φ_i can have rank 2. Thus to even consider the signature conjecture, we need to show that $\nu_3(R)$ is well defined, that is, that any R may be expressed as a linear combination of $\{R_{\varphi_i}\}$ where $\text{Rank}(\varphi_i) \geq 3$. It is also useful to check that $\nu_k(R)$ is well defined, as a higher rank requirement is one way to strengthen the conjecture, which we address later. In this section, we show that $\nu_k(R)$ is well defined for any R and any $k \in 3,\ldots,n$, and we provide an upper bound on $\nu_k(R)$.

Theorem 2.1 *$\nu_k(R) \leq 2\nu_{k-1}(R)$ for any $R \in \mathcal{A}(V)$ and any $k \in 3,\ldots,n$.*

Proof. We prove this by induction. Choose any $R \in \mathcal{A}(V)$. We may express $R = \sum \epsilon_i R_{\varphi_i}$ where $\text{Rank}(\varphi_i) = 2$ [4]. Since φ_i is symmetric, we may choose some basis where

$$\varphi_i = \text{diag}(\lambda_1, \lambda_2, 0, \ldots, 0).$$

Define

$$A_i = \text{diag}(\frac{\lambda_1}{\sqrt{2}}, \frac{\lambda_2}{\sqrt{2}}, 1, 0, \ldots, 0,) \text{ and } B_i = \text{diag}(\frac{\lambda_1}{\sqrt{2}}, \frac{\lambda_2}{\sqrt{2}}, -1, 0, \ldots, 0).$$

The only entries of $R_{A_i} + R_{B_i}$ which could be non-zero are determined by $(R_{A_i} + R_{B_i})_{1221}$, $(R_{A_i} + R_{B_i})_{1331}$, and $(R_{A_i} + R_{B_i})_{2332}$. Similarly, R_{φ_i} is completely determined by $(R_{A_i} +$

$R_{B_i})_{1221}$. Calculating each entry,

$$(R_{A_i} + R_{B_i})_{1221} = \frac{\lambda_1}{\sqrt{2}} \cdot \frac{\lambda_2}{\sqrt{2}} + \frac{\lambda_1}{\sqrt{2}} \cdot \frac{\lambda_2}{\sqrt{2}} = \lambda_1\lambda_2 = (R_{\varphi_i})_{1221}$$
$$(R_{A_i} + R_{B_i})_{1331} = \frac{\lambda_1}{\sqrt{2}} - \frac{\lambda_1}{\sqrt{2}} = 0 = (R_{\varphi_i})_{1331}$$
$$(R_{A_i} + R_{B_i})_{2332} = \frac{\lambda_2}{\sqrt{2}} - \frac{\lambda_2}{\sqrt{2}} = 0 = (R_{\varphi})_{2332}.$$

Thus $R_{\varphi_i} = R_{A_i} + R_{B_i}$ where $\mathrm{Rank}(A_i) = \mathrm{Rank}(B_i) = 3$.

Define $A_i = \psi_{2i}$ and $B_i = \psi_{2i+1}$. Repeating this process for each i, we find that $R = \sum \epsilon_i R_{\psi_i}$ where $\mathrm{Rank}(\psi_i) = 3$. Since there are at most $\nu_2(R) = \nu(R)$ φ_i and each R_{φ_i} is replaced with 2 R_{ψ_j}, $\nu_3(R) \leq \nu_2(R)$.

Let $R = \sum \epsilon_i R_{\varphi_i}$ where $\mathrm{Rank}(\varphi_i) = k-1$ for some k with $2 \leq k-1 < n$. For each φ_i, there is some basis where

$$\varphi_i = \mathrm{diag}(0, \ldots, 0, \lambda_1, \ldots, \lambda_{k-1})$$

for $\lambda_i \in \mathbb{R}$. Define

$$A_i = \mathrm{diag}\Big(0, \ldots, 0, 1, \frac{\lambda_1}{\sqrt{2}}, \ldots, \frac{\lambda_{k-1}}{\sqrt{2}}\Big) \text{ and}$$
$$B_i = \mathrm{diag}\Big(0, \ldots, 0, -1, \frac{\lambda_1}{\sqrt{2}}, \ldots, \frac{\lambda_{k-1}}{\sqrt{2}}\Big).$$

One can check that $R_{\varphi_i} = R_{A_i} + R_{B_i}$. Let the number of diagonal entries equal to 0 in φ_i be s. If i or $j \leq s$, $(R_{\varphi_i})_{ijji} = 0$, and if i and $j > s$, $(R_{\varphi_i})_{ijji} = \lambda_i\lambda_j$. Then if i or $j \leq s-1$,

$$(R_{A_i} + R_{B_i})_{ijji} = 0 = (R_{\varphi_i})_{ijji}.$$

If i and $j > s$,

$$(R_{A_i} + R_{B_i})_{ijji} = \frac{\lambda_i\lambda_j}{2} + \frac{\lambda_i\lambda_j}{2} = \lambda_i\lambda_j = (R_{\varphi_i})_{ijji}.$$

Finally, if $j > s$,

$$(R_{A_i} + R_{B_i})_{sjjs} = \frac{\lambda_j}{\sqrt{2}} - \frac{\lambda_j}{\sqrt{2}} = 0 = (R_{\varphi_i})_{sjjs}.$$

Thus $R_{\varphi_i} = R_{A_i} + R_{B_i}$.

Define $A_i = \psi_{2i}$ and $B_i = \psi_{2i+1}$. Repeating the process for each i, we see that $R = \sum \epsilon_i R_{\psi_i}$ where $\mathrm{Rank}(\psi_i) = k$. By induction, for any choice of k where $2 \leq k \leq n$, any R can be written $R = \sum \epsilon_i R_{\psi_i}$ where $\mathrm{Rank}(\psi_i) = k$. Then $\nu_k(R)$ is well defined for all such k, and since there are at most $\nu_{k-1}(R)$ R_{φ_i} to be replaced in moving from a rank $k-1$ expression of R to a rank k expression, R can be expressed as a sum of at most $2\nu_{k-1}(R)$ forms of rank k. □

Corollary 2.2 $\nu_k(n) \leq 2\nu_{k-1}(n)$ *for any* $k \in 3,\ldots,n$.

Proof. By definition, $\nu_{k-1}(R) \leq \nu_{k-1}(n)\ \forall R$. The theorem shows that

$$\nu_k(R) \leq 2\nu_{k-1}(R) \leq 2\nu_{k-1}(n)$$

for all R, so it is clear that $\nu_k(n) \leq 2\nu_{k-1}(n)$. □

Corollary 2.3 $\nu_k(n) \leq 2^{k-3}n(n+1)$ *for any* $k \in 2,\ldots,n$.

Proof. In [4], it was shown that $\nu_2(n) = \nu(n) \leq \frac{n(n+1)}{2}$. When $k = 3$, $2^{k-3} = 1$. The previous theorem shows that $\nu_3(n) \leq 2\nu_2(n) = 2\nu(n) \leq n(n+1)$. If $\nu_k(n) \leq 2^{k-3}n(n+1)$ for some k, then the theorem implies $\nu_{k+1}(n) \leq 2\nu_k(n) \leq 2^{k-2}n(n+1)$. Thus the corollary is true by induction. □

The following theorem demonstrates that in at least some cases, $\nu_k(n) < 2\nu_{k-1}(n)$.

Theorem 2.4 $\nu_3(3) = \nu(3) = 2$

Proof. In [2], it was shown that $\nu(3) = 2$. It was also shown that when $dim(V) = 3$, any $R \in \mathcal{A}(V)$ is exactly one of the following: $R = R_\varphi$ where $\text{Rank}(\varphi) = 3$, $R = R_\varphi$ where $\text{Rank}(\varphi) = 2$, or $R = R_{\varphi_1} + R_{\varphi_2}$ and $R \neq R_\varphi$ for any φ where, on some basis,

$$\varphi_1 = diag(0, 1, \lambda_2) \text{ and } \varphi_2 = diag(1, 0, \lambda_1) \text{ for some nonzero } \lambda_i.$$

In the first case, $\nu_3(R) = 1$. In the second case, Gilkey showed in [4] that $R_\varphi \neq R_\psi$ for any φ with rank 2 and ψ with rank 3, so $\nu_3(R) \neq 1$. There is some basis where $R = diag(0, a, b)$. Then, using Theorem 2.1, $R = R_A + R_B$ for $A = diag\big(1, \frac{a}{\sqrt{2}}, \frac{b}{\sqrt{2}}\big)$ and $B = diag\big(-1, \frac{a}{\sqrt{2}}, \frac{b}{\sqrt{2}}\big)$, so $\nu_3(R) = 2$.
In the third case, it is again clear that $\nu_3(R) > 1$, but one can check that $R_{\varphi_1} + R_{\varphi_2} = R_{\tau_1} + R_{\tau_2}$ where

$$\tau_1 = \text{diag}\Big(\frac{1}{\sqrt{3}}, -\sqrt{3}, \frac{\sqrt{3}\lambda}{2}\Big) \text{ and } \tau_2 = \text{diag}\Big(1, 1, \frac{\lambda}{2}\Big) \text{ if } \lambda = \lambda_1 = -\lambda_2,$$

$$\tau_1 = \text{diag}\Big(\frac{1}{\sqrt{3}}, \sqrt{3}, \frac{\sqrt{3}\lambda}{2}\Big) \text{ and } \tau_2 = \text{diag}\Big(1, -1, \frac{\lambda}{2}\Big) \text{ if } \lambda = \lambda_1 = \lambda_2,$$

and

$$\tau_1 = \text{diag}\Big(\sqrt{2}, \sqrt{2}, \frac{\lambda_1 + \lambda_2}{\sqrt{8}}\Big) \text{ and } \tau_2 = \text{diag}\Big(-\sqrt{2}, \sqrt{2}, \frac{\lambda_1 - \lambda_2}{\sqrt{8}}\Big)$$

otherwise. For any nonzero choice of λ_i, $\text{Rank}(\tau_i) = 3$, so $\nu_3(R_{\varphi_1} + R_{\varphi_2}) = 2$. Thus $\nu_3(3) = 2$. □

3 Counterexamples to the Signature Conjecture

In the original statement of the signature conjecture, we require that any expression of R uses forms of at least rank 3. To generate a counterexample, choose any real numbers a and b with $|b| > |a|$. Define R_τ where

$$\tau = \text{diag}\Big(0, \ldots, 0, \sqrt{b^2 - a^2}, \sqrt{b^2 - a^2}\Big).$$

This is a counterexample, since $R_\tau = R_A + R_B = R_{\bar{A}} - R_{\bar{B}}$ where

$$A = \operatorname{diag}\left(0, \dots, 0, 1, \frac{\sqrt{b^2 - a^2}}{\sqrt{2}}, \frac{\sqrt{b^2 - a^2}}{\sqrt{2}}\right),$$
$$B = \operatorname{diag}\left(0, \dots, 0, -1, \frac{\sqrt{b^2 - a^2}}{\sqrt{2}}, \frac{\sqrt{b^2 - a^2}}{\sqrt{2}}\right),$$

$$\bar{A} = \operatorname{diag}\big(0, \dots, 0, a, b, b\big), \text{ and } \bar{B} = \operatorname{diag}\big(0, \dots, 0, b, a, a\big).$$

One might attempt to revise the Signature Conjecture by requiring a higher minimal rank for each of the forms involved:

$$R = \sum_{i=1}^{\nu_k(R)} \epsilon_i R_{\varphi_i}, \text{ where } \operatorname{Rank}(\varphi_i) \geq k,$$

but this revision again fails, as the following theorem shows.

Theorem 3.1 *For any symmetric bilinear form τ with rank $k - 1 \leq n - 1$, $R_\tau = R_A + R_B = R_{\bar{A}} - R_{\bar{B}}$ for some symmetric bilinear forms A, B, $\bar{A}$, and $\bar{B}$ with rank k.*

Proof. Take any symmetric bilinear form τ of signature $(p, q, s+1)$ where $p + q = k - 1$. We can find a basis where

$$\tau = \operatorname{diag}(\underbrace{0, \dots, 0}_{s+1}, \underbrace{-1, \dots, -1}_{p}, \underbrace{1, \dots, 1}_{q}).$$

Then one can check that $R_\tau = R_A + R_B$ for

$$A = \operatorname{diag}\Big(\underbrace{0, \dots, 0}_{s}, 1, \underbrace{\frac{-1}{\sqrt{2}}, \dots, \frac{-1}{\sqrt{2}}}_{p}, \underbrace{\frac{1}{\sqrt{2}}, \dots, \frac{1}{\sqrt{2}}}_{q}\Big),$$
$$B = \operatorname{diag}\Big(\underbrace{0, \dots, 0}_{s}, -1, \underbrace{\frac{-1}{\sqrt{2}}, \dots, \frac{-1}{\sqrt{2}}}_{p}, \underbrace{\frac{1}{\sqrt{2}}, \dots, \frac{1}{\sqrt{2}}}_{q}\Big)$$

and $R = R_{\bar{A}} - R_{\bar{B}}$ for

$$\bar{A} = \operatorname{diag}(\underbrace{0, \dots, 0}_{s}, a, \underbrace{-b, \dots, -b}_{p}, \underbrace{b, \dots, b}_{q}), \text{ and}$$
$$\bar{B} = \operatorname{diag}(\underbrace{0, \dots, 0}_{s}, b, \underbrace{-a, \dots, -a}_{p}, \underbrace{a, \dots, a}_{q})$$

where $b = \frac{1}{a}$ and $\frac{1}{a^2} - a^2 = 1$. In other words, $a = \pm\sqrt{\frac{\sqrt{5}-1}{2}} = \pm\frac{1}{\sqrt{\varphi}}$ where φ is the golden ratio. Put differently, if

$$T_1 = \text{diag}(\underbrace{0,\ldots,0}_{s}, \frac{1}{\varphi}, \underbrace{-1,\ldots,-1}_{p}, \underbrace{1,\ldots,1}_{q}), \text{ and}$$
$$T_2 = \text{diag}(\underbrace{0,\ldots,0}_{s}, \varphi, \underbrace{-1,\ldots,-1}_{p}, \underbrace{1,\ldots,1}_{q}),$$

$R_{\bar{A}} = R_{\sqrt{\varphi}T_1}$ and $R_{\bar{B}} = R_{\frac{1}{\sqrt{\varphi}}T_2}$, so $R = \varphi R_{T_1} - \frac{1}{\varphi}R_{T_2}$.

□

Counterexamples of this type can be avoided in a possible revision to the Signature Conjecture by requiring that $\nu(R) = \nu_k(R)$ for a chosen minimal rank k.

Definition 3.2 *An algebraic curvature tensor R is absolutely minimal in rank k if $\nu_k(R) = \nu(R)$.*

The above counterexamples demonstrate that absolute minimality is necessary for a reasonable restatement of the Signature Conjecture. The following result demonstrates that it is not sufficient when $k = 3$.

Theorem 3.3 *Let $dim(V) = 3$. There exists an algebraic curvature tensor R such that $\nu(R) = 2$ and $R = R_{\tau_1} + R_{\tau_2} = R_{\psi_1} - R_{\psi_2}$ for some symmetric bilinear forms τ_1, τ_2, ψ_1, and ψ_2, all with rank 3.*

Proof. Let $R = R_{\varphi_1} + R_{\varphi_2}$ where

$$\varphi_1 = \text{diag}(0, 1, \lambda_1) \text{ and } \varphi_2 = \text{diag}(1, 0, \lambda_2) \text{ with } \lambda_i \neq 0$$

for some nonzero λ_1 and λ_2. By [2], $\nu(R) = 2$. $R_{\psi_1} - R_{\psi_2}$ for

$$\psi_1 = \text{diag}(\lambda_1, \lambda_2, 2) \text{ and } \psi_2 = \text{diag}(\lambda_1, \lambda_2, 1)$$

and $R = R_{\tau_1} + R_{\tau_2}$ where τ_1 and τ_2 are defined as in the proof of Theorem 2.4. Since λ_1 and λ_2 were chosen to be nonzero,

$$\text{Rank}(\tau_1) = \text{Rank}(\tau_2) = \text{Rank}(\psi_1) = \text{Rank}(\psi_2) = 3.$$

□

Corollary 3.4 *For any positive integer n, there exists an algebraic curvature tensor $R \in \mathcal{A}(V)$ where $dim(V) = n$ such that $\nu(R) = 2$ and $R = R_{\tau_1} + R_{\tau_2} = R_{\psi_1} - R_{\psi_2}$ for some symmetric bilinear forms τ_1, τ_2, ψ_1, and ψ_2 with rank at least 3.*

Proof. Let $R = R_{\varphi_1} + R_{\varphi_2}$ where

$$\varphi_1 = \text{diag}(\underbrace{0,\ldots,0}_{n-2}, 1, \lambda_1) \text{ and } \varphi_2 = \text{diag}(\underbrace{0,\ldots,0}_{n-3}, 1, 0, \lambda_2) \text{ with } \lambda_i \neq 0$$

for some nonzero λ_1 and λ_2. The proof that $\nu(R) = 2$ given in [2] still holds when we extend R to dimension n by adding more 0 entries on the diagonal, so $\nu(R) = 2$. Following the proof of Theorem 2.4, $R = R_{\psi_1} - R_{\psi_2}$ where

$$\psi_1 = \text{diag}(\underbrace{0, \dots, 0}_{n-3}, \lambda_1, \lambda_2, 2) \text{ and } \psi_2 = \text{diag}(\underbrace{0, \dots, 0}_{n-3}, \lambda_1, \lambda_2, 1),$$

and $R = R_{\tau_1} + R_{\tau_2}$ where if $\lambda = \lambda_1 = -\lambda_2 \neq 0$, then

$$\tau_1 = \text{diag}\Big(\underbrace{0, \dots, 0}_{n-3}, \frac{1}{\sqrt{3}}, -\sqrt{3}, \frac{3\lambda}{2\sqrt{3}}\Big) \text{ and } \tau_2 = \text{diag}\Big(\underbrace{0, \dots, 0}_{n-3}, 1, 1, \frac{\lambda}{2}\Big),$$

if $\lambda = \lambda_1 = \lambda_2 \neq 0$, then

$$\tau_1 = \text{diag}\Big(\underbrace{0, \dots, 0}_{n-3}, \frac{1}{\sqrt{3}}, \sqrt{3}, \frac{3\lambda}{2\sqrt{3}}\Big) \text{ and } \tau_2 = \text{diag}\Big(\underbrace{0, \dots, 0}_{n-3}, 1, -1, \frac{\lambda}{2}\Big),$$

and if $|\lambda_1| \neq |\lambda_2|$, then

$$\tau_1 = \text{diag}\Big(\underbrace{0, \dots, 0}_{n-3}, \sqrt{2}, \sqrt{2}, \frac{\lambda_1 + \lambda_2}{\sqrt{8}}\Big) \text{ and } \tau_2 = \text{diag}\Big(\underbrace{0, \dots, 0}_{n-3}, -\sqrt{2}, \sqrt{2}, \frac{\lambda_1 - \lambda_2}{\sqrt{8}}\Big).$$

□

4 Revisions to the Signature Conjecture

Since all the absolutely minimal counterexamples have $k = 3$, it may be sufficient to require $k \geq 4$. A revised signature conjecture would then be:

Conjecture 4.1 Given an expression $R = \sum_{i=1}^{\nu(R)} \alpha_i R_{\varphi_i}$ where $\alpha_i = \pm 1$ and $\text{Rank}(\varphi_i) \geq 4$, the number of i for which $\alpha_i = -1$ is unique.

The simplest form of a counterexample to this revised signature conjecture would be any R such that $\nu(R) = 2$ and $R = R_{\tau_1} + R_{\tau_2} = R_{\psi_1} - R_{\psi_2}$ for some τ_i and ψ_i with rank at least k for some $k \geq 4$. We are not aware of any examples fitting these criteria.

In every counterexample we have demonstrated for $k > 3$, the signatures of the symmetric bilinear forms involved in an expression of R differ when the signs involved differ. We cannot simply require that the multiset of signatures of the φ_i is equal to the multiset of signatures of the ψ_j in any two absolutely minimal expressions $R = \sum_{i=1}^{\nu(R)} \alpha_i R_{\varphi_i} = \sum_{j=1}^{\nu(R)} \epsilon_j R_{\psi_j}$ where $Rank(\varphi_i) = Rank(\psi_j) = n$ in dimension 4 or higher; we must account for the fact that $R_\varphi = R_{-\varphi}$ and the signatures of φ and $-\varphi$ can differ: if the signature of φ is (p, q, s), the signature of $-\varphi$ is (q, p, s). This leads to the definition of an adjusted signature of φ and another possible revision of the signature conjecture.

Definition 4.2 *The adjusted signature of a bilinear form φ is the signature (p, q, s) of φ if $q \geq p$ and the signature (q, p, s) of $-\varphi$ if $p > q$.*

Conjecture 4.3 In any two absolutely minimal expressions in dimension $n \geq 4$, $R = \sum_{i=1}^{\nu(R)} \alpha_i R_{\varphi_i} = \sum_{j=1}^{\nu(R)} \epsilon_j R_{\psi_j}$ where $\text{Rank}(\varphi_i) = \text{Rank}(\psi_j) = n$ and the multiset of adjusted signatures of the φ_i is equal to the multiset of adjusted signatures of the ψ_j, the number of i for which $\alpha_i = -1$ is equal to the number of j for which $\epsilon_j = -1$.

We consider only $k \geq 4$ because the case $R = R_{\varphi_1} + R_{\varphi_2}$ where $\varphi_1 = \text{diag}(0, \ldots, 0, 1, \lambda)$, $\varphi_2 = \text{diag}(0, \ldots, 0, 1, 0, \lambda)$, and $\lambda < 0$ is a counterexample if $k = 3$. This can be seen by checking the signatures of the rank 3 τ_i and ψ_i defined in the previous section such that $R = R_{\tau_1} + R_{\tau_2} = R_{\psi_1} - R_{\psi_2}$, as in Corollary 4.2.1.

5 Future Work

1. What is the nature of all counterexamples to the signature conjecture as originally stated? Does there exist an R in dimension 4 or higher for which $\nu(R) = 2$, $R = R_{\tau_1} + R_{\tau_2}$ for some τ_i with rank n, and $R = R_{\psi_1} - R_{\psi_2}$ for some ψ_i with rank n?

2. In the dimension 3 case, it was shown that $\nu_3(3) = \nu(3) = 2$, so $\nu_3(3) < 2\nu_2(3) = 4$. Can the bounds on $\nu_k(n)$ be improved upon in other cases?

3. When does $R_\varphi = R_{\tau_1} + R_{\tau_2} = R_{\psi_1} - R_{\psi_2}$ where $Rank(\varphi) = k$ and $\text{Rank}(\tau_i) = \text{Rank}(\psi_i) = k - 1$? Some cases to this are already known [6], but a more complete classification could be useful in proving one of the revised signature conjectures.

4. Given R, what is

$$\bar{\nu}_k(R) = \min_N \left\{ R = \sum_{i=1}^{N} \alpha_i R_{\varphi_i} | \text{Rank}(\varphi_i) = k \right\}?$$

   Is there a revision of the Signature Conjecture that involved φ_i of exactly rank k for some given k?

5. Which revision from Section 4, if any, of the signature conjecture holds?

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