On the Monotonicity of the Number of Positive Entries in Nonnegative Four Element Matrix Powers

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Abstract - Let A be a $m \times m$ nonnegative square matrix and let $F(A)$ denote the number of positive entries in A. We consider conditions on A to make the sequence $\{F(A^n)\}_{n=1}^{\infty}$ monotone. This is known for $F(A) \leq 3$ and $F(A) \geq m^2 - 2m + 2$; we extend this to $F(A) = 4$.

Keywords : nonnegative matrix; power; monotonicity

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1 Introduction

A nonnegative matrix is one with nonnegative real entries. Nonnegative matrices are important to study because they are applied throughout academia in areas such as economics, combinatorics, and probability; for more information, see [1]. We define F to be a function from the nonnegative square matrices to the integers that counts the number of positive entries in nonnegative square matrices. We can then classify the sequence $\{F(A^n)\}_{n=1}^{\infty}$ as non-monotonic, monotonically increasing, monotonically decreasing, or constant. In [2], Xie proved that for any $m \times m$ zero-one matrix A, if $F(A) \leq 3$ or $F(A) > m^2 - 2m + 2$ then the sequence $\{F(A^n)\}_{n=1}^{\infty}$ is monotonic. In this paper, we look at the case when $F(A) = 4$ and seek to classify all possible values for A. It is easy to see that the value of each positive matrix element provides no extra insight. Thus, we can define our matrices to be one of Boolean propositions as defined in [3] and shown below. Boolean propositions are important in mathematics and are still widely studied (e.g. [4]). These propositions can be one of two elements, unity and zero, with the operations defined as follows:

+	0	1
0	0	1
1	1	1

,

×	0	1
0	0	0
1	0	1

Then, we define E_{xy} to be a square matrix of Boolean propositions with $F(A) = 1$ and its unity element at row x, column y. For the sake of brevity, we can call these square matrices of Boolean propositions zero-one matrices. Using the above operation tables we can demonstrate that for $a, b, x, y \in \mathbb{N}$ where $\mathbb{N} = \mathbb{Z}_{>0}$, $E_{xy} + E_{ab} = E_{xy}$ if $a = x$ and $b = y$. Also, $E_{xy}E_{ab} = E_{xb}$ when $y = a$, otherwise $E_{xy}E_{ab} = 0$. Using our one element zero-one matrices, we can build square matrices of Boolean propositions with multiple elements by summing them. Multiplication of these matrices relies on distributivity: $(E_{xy} + E_{ab})^2 = E_{xy}(E_{xy} + E_{ab}) + E_{ab}(E_{xy} + E_{ab}) = E_{xy}E_{xy} + E_{xy}E_{ab} + E_{ab}E_{xy} + E_{ab}E_{ab}$.

We call the *type* of matrix A, a tuple whose k^{th} coordinate is the number of nonzero elements of row k of A (with trailing zeroes omitted). For example, the type of $E_{22} + E_{23} + E_{41}$ is $(0, 2, 0, 1)$. Now, given a matrix A, we may choose a permutation matrix P so that the type of PA is monotonically decreasing; further, the type of PA is the same as the type of PAP^{-1} because right multiplying by the permutation matrix P^{-1} only permutes columns. Note for all $n \in \mathbb{N}$, $F(A^n) = F(PA^nP^{-1}) = F((PAP^{-1})^n)$. Hence, the sequence $\{F(A^n)\}_{n=1}^{\infty}$ agrees with $\{F((PAP^{-1})^n)\}_{n=1}^{\infty}$. By replacing A with PAP^{-1} (in "standard form") we assume, without loss of generality, that A has a monotonically decreasing type henceforth. The types of standard form can only be $(1, 1, 1, 1)$, $(2, 1, 1)$, $(2, 2)$, $(3, 1)$, or (4).

Our results allow us to conclude that for zero-one matrix A with a monotonically decreasing type and $F(A) = 4$, $\{F(A^n)\}_{n=1}^{\infty}$ is monotonic unless A is one of the following matrices:

1. $A = E_{12} + E_{13} + E_{21} + E_{31}$; or
2. For some $b \in \mathbb{N}$ with $b > 3$ and for some $d \in \{1, 2\}$, $A = E_{12} + E_{1b} + E_{21} + E_{3d}$; or
3. For some $b \in \mathbb{N}$ with $b > 3$ and for some $c \in \{1, 3\}$, $A = E_{13} + E_{1b} + E_{2c} + E_{31}$.

In this first section, we establish some important terminology that will be useful in proving the monotonicity of zero-one matrices.

Definition 1.1 *Let $k, m \in \mathbb{N}$. We say that a zero-one matrix A is k-periodic starting at m if $A^m = A^{m+k}$.*

Theorem 1.2 *Let the zero-one matrix A be k-periodic starting at m for some $k, m \in \mathbb{N}$ with $F(A^m) = F(A^{m+1}) = \cdots = F(A^{m+k-1})$. Then $\{F(A^n)\}_{n=m}^{\infty}$ is constant.*

Proof. Let $n \in \mathbb{N}$ with $n \geq m$. Apply the Division Algorithm on $n - m, k$. Then, $n - m = kq + r$ for some unique $q, r \in \mathbb{Z}$ with $0 \leq r < k$. Hence, $n = kq + m + r$ and thus $A^n = A^{kq+m+r} = A^{m+r}$ by periodicity. Therefore, $F(A^n) = F(A^m)$ and so $\{F(A^n)\}_{n=m}^{\infty}$ is constant. □

Definition 1.3 *Let $k > 0$. We say that a zero-one matrix A is k-stable if A is 1-periodic starting at k.*

Corollary 1.4 *Let A be a k-stable zero-one matrix. Then, $\{F(A^n)\}_{n=k}^{\infty}$ is constant.*

Proof. This follows immediately from Theorem 1.2. □

2 Type (1,1,1,1) and type (m) for every positive m

First, we view the monotonically decreasing types $(1, 1, 1, 1)$ and (4). We will show that for any zero-one matrix A with types $(1, 1, \ldots, 1)$ or (m) where m is nonnegative, the sequence $\{F(A^n)\}_{n=1}^{\infty}$ is monotonic.

Proposition 2.1 *Let A, B be zero-one matrices with at most one 1 in each row. Then, AB maintains having at most one 1 in each row and $F(AB) \leq F(A)$.*

Proof. Let C, D be the sets of one element zero-one matrices that sum to A, B respectively. For each $e \in C$, there can only be at most one $f \in D$ such that $ef \neq 0$ since each row of B has at most one 1. Therefore, $F(AB) \leq F(A)$. Since ef maintains the row of e and each $e \in C$ has a unique row, AB has at most one 1 in each row. ☐

Corollary 2.2 *Let the zero-one matrix A have monotonically decreasing type* $(1, 1, \ldots, 1)$. *Then $\{F(A^n)\}_{n=1}^{\infty}$ is monotonically decreasing.*

Proof. We prove by way of induction.
Base Case: By Proposition 2.1, A^2 has at most one 1 in each row and $F(A^2) \leq F(A)$.
Inductive Case: Assume $n > 1$, A^n has at most one 1 in each row, and $F(A^n) \leq F(A^{n-1})$.
By Proposition 2.1, A^{n+1} has at most one 1 in each row and $F(A^{n+1}) \leq F(A^n)$.
As such, the sequence $\{F(A^n)\}_{n=1}^{\infty}$ is monotonically decreasing. ☐

Corollary 2.3 *Let the zero-one matrix A have type (m) for some $m \in \mathbb{N}$.*
Then, $\{F(A^n)\}_{n=1}^{\infty}$ is monotonically decreasing.

Proof. Applying Corollary 2.2 to A^T, we get $\{F((A^T)^n)\}_{n=1}^{\infty}$ is monotonically decreasing, therefore so is $\{F(A^n)\}_{n=1}^{\infty}$. ☐

3 Type (2,2)

We now look at the case in which zero-one matrix A has monotonically decreasing type $(2, 2)$. In this case, some matrices exist such that $\{F(A^n)\}_{n=1}^{\infty}$ is monotonically decreasing and others exist such that $\{F(A^n)\}_{n=1}^{\infty}$ is monotonically increasing. Another way this section differs from the previous is in the use of terminology defined in section 1. We use the below table and following propositions to prove Theorem 3.1. Since A has type $(2, 2)$, we say that $a < b$ and $c < d$. The cases are based on a, b, and c and the subcases are based on d.

Case 1. $c < a < b$.

- Subcase 1a. $c < d < a < b$ is covered by Propositions 3.3, 3.5, 3.6 and 3.7.
- Subcase 1b. $c < a = d < b$ is covered by Propositions 3.3, 3.5, 3.6, and 3.11.
- Subcase 1c. $c < a < d < b$ is covered by Propositions 3.3, 3.5, 3.6 and 3.10.
- Subcase 1d. $c < a < b = d$ is covered by Propositions 3.3, 3.5, 3.6, and 3.10.
- Subcase 1e. $c < a < b < d$ is covered by Propositions 3.3, 3.5, 3.6, and 3.10.

Case 2. $a = c < b$.

Subcase 2a. $a = c < d < b$ is covered by Propositions 3.3, 3.4, and 3.8.
Subcase 2b. $a = c < b = d$ is covered by Propositions 3.2, 3.3, and 3.8.
Subcase 2c. $a = c < b < d$ is covered by Propositions 3.3, 3.8, and 3.11.
Case 3. $a < c < b$ is covered by Propositions 3.3, 3.5, 3.6, and 3.9.
Case 4. $a < b = c < d$ is covered by Propositions 3.3, 3.4, 3.5, and 3.6.
Case 5. $a < b < c < d$ is covered by Propositions 3.3, 3.5, 3.6, and 3.7.

Theorem 3.1 *Let the zero-one matrix A have monotonically decreasing type $(2, 2)$. Then, $\{F(A^n)\}_{n=1}^{\infty}$ is monotonic.*

Proposition 3.2 *Let $A = E_{11} + E_{12} + E_{21} + E_{22}$. Then, A is 1-stable and $\{F(A^n)\}_{n=1}^{\infty}$ is constant.*

Proof. Direct calculation (matrix multiplication) shows that $A = A^2$. The rest follows immediately. □

Proposition 3.3 *Let $A = E_{1a} + E_{1b} + E_{2c} + E_{2d}$ with $a, b, c, d > 2$. Then, A is 2-stable and $\{F(A^n)\}_{n=1}^{\infty}$ is monotonically decreasing.*

Proof. Direct calculation shows that $F(A) > F(A^2)$ and $A^2 = A^3$. The rest follows immediately. □

Proposition 3.4 *Let $A = E_{11} + E_{12} + E_{22} + E_{2d}$ or $A = E_{11} + E_{1d} + E_{21} + E_{22}$ with $d > 2$. Then, A is 2-stable and $\{F(A^n)\}_{n=1}^{\infty}$ is monotonically increasing.*

Proof. Direct calculation shows that $F(A) < F(A^2)$ and $A^2 = A^3$. The rest follows immediately. □

Proposition 3.5 *Let $A = E_{11} + E_{1b} + E_{2c} + E_{2d}$ or $A = E_{1c} + E_{1d} + E_{22} + E_{2b}$ with $b, c, d > 2$. Then, A is 2-stable and $\{F(A^n)\}_{n=1}^{\infty}$ is monotonically decreasing.*

Proof. Direct calculation shows that $F(A) > F(A^2)$ and $A^2 = A^3$. The rest follows immediately. □

Proposition 3.6 *Let $A = E_{12} + E_{1b} + E_{2c} + E_{2d}$ or $A = E_{1c} + E_{1d} + E_{21} + E_{2b}$ with $b, c, d > 2$. Then, A is 3-stable and $\{F(A^n)\}_{n=1}^{\infty}$ is monotonically decreasing.*

Proof. Direct calculation shows that $F(A) > F(A^2) > F(A^3)$ and $A^3 = A^4$. The rest follows immediately. □

Proposition 3.7 *Let $A = E_{11} + E_{12} + E_{2c} + E_{2d}$ or $A = E_{1c} + E_{1d} + E_{21} + E_{22}$ with $c, d > 2$. Then, A is 2-stable and $\{F(A^n)\}_{n=1}^{\infty}$ is constant.*

Proof. Direct calculation shows that $F(A) = F(A^2)$ and $A^2 = A^3$. The rest follows immediately. □

Proposition 3.8 *Let $A = E_{11} + E_{1b} + E_{21} + E_{2d}$ or $A = E_{12} + E_{1d} + E_{22} + E_{2b}$ with $b, d > 2$. Then, A is 2-stable and $\{F(A^n)\}_{n=1}^{\infty}$ is constant.*

Proof. Direct calculation shows that $F(A) = F(A^2)$ and $A^2 = A^3$. The rest follows immediately. □

Proposition 3.9 *Let $A = E_{11} + E_{1b} + E_{22} + E_{2d}$ with $b, d > 2$. Then, A is 1-stable and $\{F(A^n)\}_{n=1}^{\infty}$ is constant.*

Proof. Direct calculation shows that $A^2 = A$. The rest follows immediately. □

Proposition 3.10 *Let $A = E_{12} + E_{1b} + E_{21} + E_{2d}$ with $b, d > 2$. Then, A is 2-periodic starting at 1 and $\{F(A^n)\}_{n=1}^{\infty}$ is constant.*

Proof. Direct calculation shows that $F(A) = F(A^2)$ and $A = A^3$. The rest follows immediately. □

Proposition 3.11 *Let $A = E_{11} + E_{12} + E_{21} + E_{2d}$ or $A = E_{12} + E_{1d} + E_{21} + E_{22}$ with $d > 2$. Then, A is 3-stable and $\{F(A^n)\}_{n=1}^{\infty}$ is monotonically increasing.*

Proof. Direct calculation shows that $F(A) < F(A^2) < F(A^3)$ and $A^3 = A^4$. The rest follows immediately. □

4 Type (2,1,1)

This case deals with matrices with monotonically decreasing type $(2, 1, 1)$. The section differs from the previous because their exists some zero-one matrix, A, where the sequence $\{F(A^n)\}_{n=1}^{\infty}$ is non-monotonic. In this section, we take each of the three element zero-one matrices with type $(2, 1)$ and add an E_{3d} to it, yielding $A = E_{1a} + E_{1b} + E_{2c} + E_{3d}$. This allows us to systematically cover each zero-one matrix with type $(2, 1, 1)$. Then, using the diagram below and following Propositions, we prove Theorem 4.1. Since A has type $(2, 1, 1)$ we assume $a < b$. The cases are based on a and the sub cases are based on b and c.

Case 1. $a = 1$.
- Subcase 1a. $b > 1, c = 1$ is covered by Proposition 4.2.
- Subcase 1b. $b > 1, c = 2$ is covered by Proposition 4.3.
- Subcase 1c. $b = 2, c > 2$ is covered by Proposition 4.4.
- Subcase 1d. $b, c > 2$ is covered by Proposition 4.5.

Case 2. $a = 2$.
- Subcase 2a. $c = 1, b > 2$ is covered by Proposition 4.6.
- Subcase 2b. $b > 2, c = 2$ is covered by Proposition 4.7.
- Subcase 2c. $b, c > 2$ is covered by Proposition 4.8.

Case 3. $a > 2$.
- Subcase 3a. $b > 3, c = 1$ is covered by Proposition 4.9.
- Subcase 3b. $b > 3, c = 2$ is covered by Proposition 4.10.
- Subcase 3c. $b > 3, c > 2$ is covered by Proposition 4.11.

Theorem 4.1 *Let $A = E_{1a} + E_{1b} + E_{2c} + E_{3d}$. Then, $\{F(A^n)\}_{n=1}^{\infty}$ is monotonic unless A is one of the following matrices:*

1. $A = E_{12} + E_{13} + E_{21} + E_{31}$; *or*

2. *For some* $b \in \mathbb{N}$ *with* $b > 3$ *and for some* $d \in \{1, 2\}$, $A = E_{12} + E_{1b} + E_{21} + E_{3d}$; *or*

3. *For some* $b \in \mathbb{N}$ *with* $b > 3$ *and for some* $c \in \{1, 3\}$, $A = E_{13} + E_{1b} + E_{2c} + E_{31}$.

Proposition 4.2 *Let* $A = E_{11} + E_{1b} + E_{21} + E_{3d}$ *with* $b > 1$. *Then,* $\{F(A^n)\}_{n=1}^{\infty}$ *is monotonic.*

Proof. Direct calculation shows the following:
When $b = 3$ and $d = 1$, $F(A) < F(A^2)$ and $A^2 = A^3$.
When $b = 3$ and $d = 2$, $F(A) \leq F(A^2) \leq F(A^3) \leq F(A^4)$ and $A^4 = A^5$.
When $b = 3$ and $d = 3$, $F(A) < F(A^2)$ and $A^2 = A^3$.
When $b = 3$ and $d > 3$, $F(A) \leq F(A^2) \leq F(A^3)$ and $A^3 = A^4$.
When $b \neq 3$ and $d = 1$, $F(A) < F(A^2)$ and $A^2 = A^3$.
When $b \neq 3$ and $d = 2$, $F(A) \leq F(A^2) \leq F(A^3)$ and $A^3 = A^4$.
When $b \neq 3$ and $d = 3$, $F(A) < F(A^2)$ and $A^2 = A^3$.
When $b \neq 3$ and $d > 3$, $F(A) = F(A^2)$ and $A^2 = A^3$.
The rest follows immediately. □

Proposition 4.3 *Let* $A = E_{11} + E_{1b} + E_{22} + E_{3d}$ *with* $b > 1$. *Then,* $\{F(A^n)\}_{n=1}^{\infty}$ *is monotonic.*

Proof. Direct calculation shows the following:
When $b = 3$ and $d = 1$, $F(A) < F(A^2)$ and $A^2 = A^3$.
When $b = 3$ and $d = 2$, $F(A) < F(A^2)$ and $A^2 = A^3$.
When $b = 3$ and $d = 3$, $A = A^2$.
When $b = 3$ and $d > 3$, $F(A) = F(A^2)$ and $A^2 = A^3$.
When $b \neq 3$ and $d = 1$, $F(A) < F(A^2)$ and $A^2 = A^3$.
When $b \neq 3$ and $d = 2$, $A = A^2$.
When $b \neq 3$ and $d = 3$, $A = A^2$.
When $b \neq 3$ and $d > 3$, $F(A^2) < F(A)$ and $A^2 = A^3$.
The rest follows immediately. □

Proposition 4.4 *Let* $A = E_{11} + E_{12} + E_{2c} + E_{3d}$ *with* $c > 2$. *Then,* $\{F(A^n)\}_{n=1}^{\infty}$ *is monotonic.*

Proof. Direct calculation shows the following:
When $c = 3$ and $d = 1$, $F(A) \leq F(A^2) \leq F(A^3) \leq F(A^4)$ and $A^4 = A^5$.
When $c = 3$ and $d = 2$, $F(A) < F(A^2) = F(A^3)$ and $A^2 = A^4$.
When $c = 3$ and $d = 3$, $F(A) < F(A^2)$ and $A^2 = A^3$.
When $c = 3$ and $d > 3$, $F(A) = F(A^2) = F(A^3)$ and $A^3 = A^4$.
When $c > 3$ and $d = 1$, $F(A) \leq F(A^2) \leq F(A^3)$ and $A^3 = A^4$.
When $c > 3$ and $d = 2$, $F(A^3) \leq F(A^2) \leq F(A)$ and $A^3 = A^4$.
When $c > 3$ and $d = 3$, $F(A) = F(A^2)$ and $A^2 = A^3$.
When $c > 3$ and $d > 3$, $F(A^2) \leq F(A)$ and $A^2 = A^3$.
The rest follows immediately. □

Proposition 4.5 *Let* $A = E_{11} + E_{1b} + E_{2c} + E_{3d}$ *with* $b, c > 2$. *Then,* $\{F(A^n)\}_{n=1}^{\infty}$ *is monotonic.*

Proof. Direct calculation shows the following:
When $b = c = 3$ and $d = 1$, $F(A) \leq F(A^2) \leq F(A^3)$ and $A^3 = A^4$.
When $b = c = 3$ and $d = 2$, $F(A) \leq F(A^2) = F(A^3)$ and $A^2 = A^4$.
When $b = c = 3$ and $d = 3$, $A = A^2$.
When $b = c = 3$ and $d > 3$, $F(A^3) \leq F(A^2) \leq F(A)$ and $A^3 = A^4$.
When $b = 3 < c$ and $d = 1$, $F(A) = F(A^2)$ and $A^2 = A^3$.
When $b = 3 < c$ and $d = 2$, $F(A) = F(A^2) = F(A^3)$ and $A^3 = A^4$.
When $b = 3 < c$ and $d = 3$, $F(A^2) < F(A)$ and $A^2 = A^3$.
When $b = 3 < c$ and $d > 3$, $F(A^2) < F(A)$ and $A^2 = A^3$.
When $c = 3 < b$ and $d = 1$, $F(A) \leq F(A^2) \leq F(A^3)$ and $A^3 = A^4$.
When $c = 3 < b$ and $d = 2$, $F(A) = F(A^2)$ and $A = A^3$.
When $c = 3 < b$ and $d = 3$, $A = A^2$.
When $c = 3 < b$ and $d > 3$, $F(A^3) \leq F(A^2) \leq F(A)$ and $A^3 = A^4$.
When $b, c > 3$ and $d = 1$, $F(A) = F(A^2)$ and $A^2 = A^3$.
When $b, c > 3$ and $d = 2$, $F(A^3) \leq F(A^2) \leq F(A)$ and $A^3 = A^4$.
When $b, c > 3$ and $d = 3$, $F(A^2) < F(A)$ and $A^2 = A^3$.
When $b, c > 3$ and $d > 3$, $F(A^2) < F(A)$ and $A^2 = A^3$.
The rest follows immediately. □

Proposition 4.6 *Let* $A = E_{12} + E_{1b} + E_{21} + E_{3d}$ *with* $b > 2$. *Then,* $\{F(A^n)\}_{n=1}^{\infty}$ *is monotonic unless* $b = 3$ *and* $d = 1$ *or* $b > 3$ *and* $d \in \{1, 2\}$.

Proof. Direct calculation shows the following:
When $b = 3$ and $d = 1$, $F(A) < F(A^2)$ and $A = A^3$.
When $b = 3$ and $d = 2$, $F(A) \leq F(A^2) \leq F(A^3) \leq F(A^4) \leq F(A^5)$ and $A^5 = A^6$.
When $b = 3$ and $d = 3$, $F(A) < F(A^2) = F(A^3)$ and $A^2 = A^4$.
When $b = 3$ and $d > 3$, $F(A) = F(A^2) = F(A^3)$ and $A^2 = A^4$.
When $b > 3$ and $d = 1$, $F(A) < F(A^2)$ and $A = A^3$.
When $b > 3$ and $d = 2$, $F(A) = F(A^2) < F(A^3)$ and $A^2 = A^4$.
When $b > 3$ and $d = 3$, $F(A) = F(A^2)$ and $A = A^3$.
When $b > 3$ and $d > 3$, $F(A^3) = F(A^2) < F(A)$ and $A^2 = A^4$.
The rest follows immediately. □

Proposition 4.7 *Let* $A = E_{12} + E_{1b} + E_{22} + E_{3d}$ *with* $b > 2$. *Then,* $\{F(A^n)\}_{n=1}^{\infty}$ *is monotonic.*

Proof. Direct calculation shows the following:
When $b = 3$ and $d = 1$, $F(A) < F(A^2) = F(A^3)$ and $A^2 = A^4$.
When $b = 3$ and $d = 2$, $F(A^2) < F(A)$ and $A^2 = A^3$.
When $b = 3$ and $d = 3$, $A = A^2$.
When $b = 3$ and $d > 3$, $F(A^3) \leq F(A^2) \leq F(A)$ and $A^3 = A^4$.
When $b > 3$ and $d = 1$, $F(A^3) \leq F(A^2) \leq F(A)$ and $A^3 = A^4$.

When $b > 3$ and $d = 2$, $F(A^2) < F(A)$ and $A^2 = A^3$.
When $b > 3$ and $d = 3$, $F(A^2) < F(A)$ and $A^2 = A^3$.
When $b > 3$ and $d > 3$, $F(A^2) < F(A)$ and $A^2 = A^3$.
The rest follows immediately. □

Proposition 4.8 *Let* $A = E_{12} + E_{1b} + E_{2c} + E_{3d}$ *with* $b, c > 2$. *Then,* $\{F(A^n)\}_{n=1}^{\infty}$ *is monotonic.*

Proof. Direct calculation shows the following:
When $b = c = 3$ and $d = 1$, $F(A) \leq F(A^2) \leq F(A^3) \leq F(A^4) \leq F(A^5)$ and $A^5 = A^6$.
When $b = c = 3$ and $d = 2$, $F(A) = F(A^2)$ and $A = A^3$.
When $b = c = 3$ and $d = 3$, $F(A^2) < F(A)$ and $A^2 = A^3$.
When $b = c = 3$ and $d > 3$, $F(A^4) \leq F(A^3) \leq F(A^2) \leq F(A)$ and $A^4 = A^5$.
When $b = 3 < c$ and $d = 1$, $F(A) = F(A^2) = F(A^3)$ and $A^2 = A^4$.
When $b = 3 < c$ and $d = 2$, $F(A^4) \leq F(A^3) \leq F(A^2) \leq F(A)$ and $A^4 = A^5$.
When $b = 3 < c$ and $d = 3$, $F(A^3) \leq F(A^2) \leq F(A)$ and $A^3 = A^4$.
When $b = 3 < c$ and $d > 3$, $F(A^3) \leq F(A^2) \leq F(A)$ and $A^3 = A^4$.
When $c = 3 < b$ and $d = 1$, $F(A) = F(A^2) = F(A^3)$ and $A = A^4$.
When $c = 3 < b$ and $d = 2$, $F(A^3) = F(A^2) < F(A)$ and $A^2 = A^4$.
When $c = 3 < b$ and $d = 3$, $F(A^2) < F(A)$ and $A^2 = A^3$.
When $c = 3 < b$ and $d > 3$, $F(A^4) \leq F(A^3) \leq F(A^2) \leq F(A)$ and $A^4 = A^5$.
When $b, c > 3$ and $d = 1$, $F(A^4) \leq F(A^3) \leq F(A^2) \leq F(A)$ and $A^4 = A^5$.
When $b, c > 3$ and $d = 2$, $F(A^3) \leq F(A^2) \leq F(A)$ and $A^3 = A^4$.
When $b, c > 3$ and $d = 3$, $F(A^3) \leq F(A^2) \leq F(A)$ and $A^3 = A^4$.
When $b, c > 3$ and $d > 3$, $F(A^3) \leq F(A^2) \leq F(A)$ and $A^3 = A^4$.
The rest follows immediately. □

Proposition 4.9 *Let* $A = E_{1a} + E_{1b} + E_{21} + E_{3d}$ *with* $a > 2$ *and* $b > 3$. *Then,* $\{F(A^n)\}_{n=1}^{\infty}$ *is monotonic unless* $a = 3$ *and* $d = 1$.

Proof. Direct calculation shows the following:
When $a = 3$ and $d = 1$, $F(A) < F(A^2)$ and $A = A^3$.
When $a = 3$ and $d = 2$, $F(A) = F(A^2) = F(A^3)$ and $A = A^4$.
When $a = 3$ and $d = 3$, $F(A^3) \leq F(A^2) \leq F(A)$ and $A^3 = A^4$.
When $a = 3$ and $d > 3$, $F(A^4) \leq F(A^3) \leq F(A^2) \leq F(A)$ and $A^4 = A^5$.
When $a > 3$ and $d = 1$, $F(A^3) \leq F(A^2) \leq F(A)$ and $A^3 = A^4$.
When $a > 3$ and $d = 2$, $F(A^4) \leq F(A^3) \leq F(A^2) \leq F(A)$ and $A^4 = A^5$.
When $a > 3$ and $d = 3$, $F(A^3) \leq F(A^2) \leq F(A)$ and $A^3 = A^4$.
When $a > 3$ and $d > 3$, $F(A^3) \leq F(A^2) \leq F(A)$ and $A^3 = A^4$.
The rest follows immediately. □

Proposition 4.10 *Let* $A = E_{1a} + E_{1b} + E_{22} + E_{3d}$ *with* $a > 2$ *and* $b > 3$. *Then,* $\{F(A^n)\}_{n=1}^{\infty}$ *is monotonic.*

Proof. Direct calculation shows the following:
When $a = 3$ and $d = 1$, $F(A) = F(A^2)$ and $A = A^3$.

When $a = 3$ and $d = 2$, $F(A^2) < F(A)$ and $A^2 = A^3$.
When $a = 3$ and $d = 3$, $F(A^2) < F(A)$ and $A^2 = A^3$.
When $a = 3$ and $d > 3$, $F(A^3) \leq F(A^2) \leq F(A)$ and $A^3 = A^4$.
When $a > 3$ and $d = 1$, $F(A^3) \leq F(A^2) \leq F(A)$ and $A^3 = A^4$.
When $a > 3$ and $d = 2$, $F(A^2) < F(A)$ and $A^2 = A^3$.
When $a > 3$ and $d = 3$, $F(A^2) < F(A)$ and $A^2 = A^3$.
When $a > 3$ and $d > 3$, $F(A^2) < F(A)$ and $A^2 = A^3$.
The rest follows immediately. □

Proposition 4.11 *Let* $A = E_{1a} + E_{1b} + E_{2c} + E_{3d}$ *with* $a, c > 2$ *and* $b > 3$. *Then,* $\{F(A^n)\}_{n=1}^{\infty}$ *is monotonic unless* $a = c = 3$ *and* $d = 1$.

Proof. Direct calculation shows the following:
When $a = c = 3$ and $d = 1$, $F(A) = F(A^3) > F(A^2)$ and $A^2 = A^4$.
When $a = c = 3$ and $d = 2$, $F(A) > F(A^2) = F(A^3)$ and $A^2 = A^4$.
When $a = c = 3$ and $d = 3$, $F(A^2) < F(A)$ and $A^2 = A^3$.
When $a = c = 3$ and $d > 3$, $F(A^3) \leq F(A^2) \leq F(A)$ and $A^3 = A^4$.
When $a = 3 < c$ and $d = 1$, $F(A) > F(A^2) = F(A^3)$ and $A^2 = A^4$.
When $a = 3 < c$ and $d = 2$, $F(A^4) \leq F(A^3) \leq F(A^2) \leq F(A)$ and $A^4 = A^5$.
When $a = 3 < c$ and $d = 3$, $F(A^2) < F(A)$ and $A^2 = A^3$.
When $a = 3 < c$ and $d > 3$, $F(A^3) \leq F(A^2) \leq F(A)$ and $A^3 = A^4$.
When $c = 3 < a$ and $d = 1$, $F(A^4) \leq F(A^3) \leq F(A^2) \leq F(A)$ and $A^4 = A^5$.
When $c = 3 < a$ and $d = 2$, $F(A) > F(A^2) = F(A^3)$ and $A^2 = A^4$.
When $c = 3 < a$ and $d = 3$, $F(A^2) < F(A)$ and $A^2 = A^3$.
When $c = 3 < a$ and $d > 3$, $F(A^3) \leq F(A^2) \leq F(A)$ and $A^3 = A^4$.
When $a, c > 3$ and $d = 1$, $F(A^3) \leq F(A^2) \leq F(A)$ and $A^3 = A^4$.
When $a, c > 3$ and $d = 2$, $F(A^3) \leq F(A^2) \leq F(A)$ and $A^3 = A^4$.
When $a, c > 3$ and $d = 3$, $F(A^2) < F(A)$ and $A^2 = A^3$.
When $a, c > 3$ and $d > 3$, $F(A^2) < F(A)$ and $A^2 = A^3$.
The rest follows immediately. □

5 Concluding Theorems

Thus far, we have proven all of the propositions needed to demonstrate the desired results. Now, we will apply some of the aforementioned theorems and corollaries to demonstrate the case when zero-one matrix A has monotonically decreasing type $(3, 1)$. Using this, we will have proven each of the five possible monotonically decreasing types in enough detail to support the final theorem. This section differs from the prior because we are able to prove the last two remaining theorems without any propositions and we do not make use of specific zero-one matrices.

Theorem 5.1 *Let the zero-one matrix* A *have type* $(3, 1)$. *Then,* $\{F(A^n)\}_{n=1}^{\infty}$ *is monotonic.*

Proof. Since A has type $(3,1)$, $A = E_{1a} + E_{1b} + E_{1c} + E_{2d}$. We prove by cases.
Case 1: Assume $d \notin \{a,b,c\}$. Then, every element must be in a distinct column. By Corollary 2.3, $\{F(A^n)\}_{n=1}^{\infty}$ is monotonic.
Case 2: Assume $d \in \{a,b,c\}$. Set $B = A^T$. Let P be the permutation matrix that makes PB have type $(2,1,1)$. Then, $PBP^{-1} = E_{1a'} + E_{1b'} + E_{2a'} + E_{3a'}$. Thus, by Theorem 4.1, $\{F((PBP^{-1})^n)\}_{n=1}^{\infty}$ is monotonic and therefore so is $\{F(A^n)\}_{n=1}^{\infty}$.
As desired, in all cases $\{F(A^n)\}_{n=1}^{\infty}$ is monotonic. □

Theorem 5.2 *Let the zero-one matrix A with $F(A) = 4$ have a monotonically decreasing type. Then $\{F(A^n)\}_{n=1}^{\infty}$ is monotonic unless A is one of the following matrices.*

1. $A = E_{12} + E_{13} + E_{21} + E_{31}$*; or*
2. $A = E_{12} + E_{1b} + E_{21} + E_{3d}$ *with* $b > 3$ *and* $d \in \{1,2\}$*; or*
3. $A = E_{13} + E_{1b} + E_{2c} + E_{31}$ *with* $b > 3$ *and* $c \in \{1,3\}$*.*

Proof. We prove by cases.
Case 1: Assume A has type $(1,1,1,1)$. Then, by Corollary 2.2, $\{F(A^n)\}_{n=1}^{\infty}$ is monotonic.
Case 2: Assume A has type $(2,1,1)$. Then, by Theorem 4.1, $\{F(A^n)\}_{n=1}^{\infty}$ is monotonic unless A satisfies $A = E_{12} + E_{13} + E_{21} + E_{31}$; $A = E_{12} + E_{1b} + E_{21} + E_{3d}$ with $b > 3$ and $d \in \{1,2\}$; or $A = E_{13} + E_{1b} + E_{2c} + E_{31}$ with $b > 3$ and $c \in \{1,3\}$.
Case 3: Assume A has type $(2,2)$. Then, by Theorem 3.1, $\{F(A^n)\}_{n=1}^{\infty}$ is monotonic.
Case 4: Assume A has type $(3,1)$. Then, by Theorem 5.1, $\{F(A^n)\}_{n=1}^{\infty}$ is monotonic.
Case 5: Assume A has type (4). Then, by Corollary 2.3, $\{F(A^n)\}_{n=1}^{\infty}$ is monotonic.
In all cases, we get the desired result. □

We found that for zero-one matrix A, the sequence $\{F(A^n)\}_{n=1}^{\infty}$ is monotonic unless A is one of the matrices defined in Theorem 5.2. Our calculations have shown that every k-stable zero-one matrix A has $\{F(A^n)\}_{n=1}^{\infty}$ monotonic, but we are unable to prove this.

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