Dilated Floor Functions that Commute Sometimes

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Abstract - We explore the dilated floor function $f_a(x) = \lfloor ax \rfloor$ and its commutativity with functions of the same form. A previous paper found all a and b such that f_a and f_b commute for all real x. In this paper, we determine all x for which the functions commute for a particular choice of a and b. We calculate the proportion of the number line on which the functions commute. We determine bounds for how far away the functions can get from commuting. We solve this fully for integer a, b and partially for real a, b.

Keywords : commutativity; dilated floor functions

Mathematics Subject Classification (2010) **:** 11A99

1 Introduction

The Floor and Ceiling functions were originally coined in name and notation by Kenneth E. Iverson in his 1962 book *A Programming Language* [2]. Their applications range widely from use in real life to use in mathematical theory. For instance, the price intervals under a progressive tax system are determined by a step function based on income level. Many services utilize the ceiling function to up-charge from a partial hour to a full hour. They are used frequently in discrete mathematics to describe objects that are not continuous as well as in number theory to provide bounded inequalities as useful estimates. Formally, for any real number x, $\lfloor x \rfloor$ is the unique integer satisfying the inequality $\lfloor x \rfloor \le x < \lfloor x \rfloor + 1$.

This paper furthers the research of Lagarias, Murayama, and Richman who investigated dilated floor functions and their relationship to drawing "straight" lines on a computer screen [1]. They determined the values of $a, b \in \mathbb{R}$ such that the dilated floor functions $f_a(x)$ and $f_b(x)$ commute under composition ($\lfloor a\lfloor bx \rfloor \rfloor = \lfloor b\lfloor ax \rfloor \rfloor$ for all real x). Specifically, they found that this equality holds when a and b are equal, at least one is equal to 0, or if both are reciprocals of positive integers. We will assume that $a \neq b$ and $ab \neq 0$ henceforth, and ask the following:

1. If they do not always commute (i.e. for all x), when *do* they commute?
2. On what fraction of the domain do they commute?
3. How far away can they get from commuting?

To do this, we translate the problem into finding zeros of $F_{a,b}(x)$, defined as:

$$F_{a,b}(x) := \lfloor a\lfloor bx\rfloor\rfloor - \lfloor b\lfloor ax\rfloor\rfloor$$

which we will also refer to as $F(x)$ when a and b are fixed. It is also useful to define $Z(a,b) = \{x : \lfloor a\lfloor bx\rfloor\rfloor = \lfloor b\lfloor ax\rfloor\rfloor\}$ to be the *commuting set*, the set of all x where the dilated floor functions commute.
We define $T_{a,b}(n) = \{x : \lfloor a\lfloor bx\rfloor\rfloor = n\}$ for $n \in \mathbb{Z}$, which we will generally refer to as a T interval. It was proved in [1] that[1]:

$$T_{a,b}(n) = \begin{cases} \left[\frac{1}{b}\lceil\frac{n}{a}\rceil, \frac{1}{b}\lceil\frac{n+1}{a}\rceil\right) & a>0, b>0 \\ \left(\frac{1}{b}(\lfloor\frac{n}{a}\rfloor+1), \frac{1}{b}(\lfloor\frac{n+1}{a}\rfloor+1)\right] & a<0, b<0 \\ \left(\frac{1}{b}\lceil\frac{n+1}{a}\rceil, \frac{1}{b}\lceil\frac{n}{a}\rceil\right] & a>0, b<0 \\ \left[\frac{1}{b}\left(\lfloor\frac{n+1}{a}\rfloor+1\right), \frac{1}{b}\left(\lfloor\frac{n}{a}\rfloor+1\right)\right) & a<0, b>0 \end{cases}$$

Since the codomain of $F(x)$ is $\mathbb{Z}$, we have $Z(a,b) = \bigcup_{n\in\mathbb{Z}} T_{a,b}(n) \cap T_{b,a}(n)$.
Another interesting notion is the proportion of the real number line such that $F_{a,b}(x) = 0$. We will call this the *commuting proportion*, which is defined as

$$P(a,b) = \lim_{N\to\infty} \frac{\mu(Z(a,b)\cap[-N,N])}{2N},$$

where μ denotes Lebesgue measure. Note that $T_{a,b}(n) \cap T_{b,a}(n)$ is always an interval, so $Z(a,b)\cap[-N,N]$ is a finite union of intervals, so we can compute $P(a,b)$ using only interval lengths. Also, by definition we have $Z(a,b) = Z(b,a)$, and $P(a,b) = P(b,a)$. We will call any interval where $F_{a,b}(x) = 0$ a *zero interval*.

Theorem 1.1 *Fix $a, b \in \mathbb{R}$, then* $Z(a,b) =$

$$\begin{cases} \bigcup_{n\in\mathbb{Z}} \left[\frac{1}{b}\lceil\frac{n}{a}\rceil, \frac{1}{b}\lceil\frac{n+1}{a}\rceil\right) \cap \left[\frac{1}{a}\lceil\frac{n}{b}\rceil, \frac{1}{a}\lceil\frac{n+1}{b}\rceil\right) & a>0, b>0 \\ \bigcup_{n\in\mathbb{Z}} \left(\frac{1}{b}(\lfloor\frac{n}{a}\rfloor+1), \frac{1}{b}(\lfloor\frac{n+1}{a}\rfloor+1)\right] \cap \left(\frac{1}{a}(\lfloor\frac{n}{b}\rfloor+1), \frac{1}{a}(\lfloor\frac{n+1}{b}\rfloor+1)\right] & a<0, b<0 \\ \bigcup_{n\in\mathbb{Z}} \left[\frac{1}{a}(\lfloor\frac{n+1}{b}\rfloor+1), \frac{1}{b}\lceil\frac{n}{a}\rceil\right] & a>0, b<0 \\ \bigcup_{n\in\mathbb{Z}} \left[\frac{1}{b}(\lfloor\frac{n+1}{a}\rfloor+1), \frac{1}{a}\lceil\frac{n}{b}\rceil\right] & a<0, b>0 \end{cases}$$

Proof. Case 1 and 2 follow from the definitions and cannot be simplified for general a and b. In case 3 and 4 we can simplify the expression to be one set instead of an intersection of two.
Without loss of generality, suppose $a > 0, b < 0$. We have that $T_{a,b}(n) \cap T_{b,a}(n) = \left(\frac{1}{b}\lceil\frac{n+1}{a}\rceil, \frac{1}{b}\lceil\frac{n}{a}\rceil\right] \cap \left[\frac{1}{a}\lfloor\frac{n+1}{b}\rfloor + \frac{1}{a}, \frac{1}{a}\lfloor\frac{n}{b}\rfloor + \frac{1}{a}\right)$. To simplify the interval, it will be enough to show that $\frac{1}{b}\lceil\frac{n}{a}\rceil < \frac{1}{a}(\lfloor\frac{n}{b}\rfloor + 1)$ for all $n \in \mathbb{Z}$. If this inequality holds for all integers n, it holds for $n+1$, and so our bounds for the interval follow.
By definition, $\frac{n}{b} - 1 < \lfloor\frac{n}{b}\rfloor \leq \frac{n}{b}$ and $\frac{n}{a} \leq \lceil\frac{n}{a}\rceil < \frac{n}{a} + 1$. Multiplying through by b and a respectively, we get $n \leq b\lfloor\frac{n}{b}\rfloor < n - b$, and $n \leq a\lceil\frac{n}{a}\rceil < n + a$. Adding b to the first

[1] The paper actually defined the upper level set $S_{a,b}(n) = \{x : \lfloor a\lfloor bx\rfloor\rfloor \geq n\}$. Therefore our $T_{a,b}$ is simply $S_{a,b}(n) \setminus S_{a,b}(n+1)$.

inequality gives $b(\lfloor \frac{n}{b} \rfloor + 1) < n$. Hence $b(\lfloor \frac{n}{b} \rfloor + 1) < a\lceil \frac{n}{a} \rceil$. Dividing by ab gives the desired result.

Case 4 uses the same proof structure as 3, with a and b swapped. $\square$

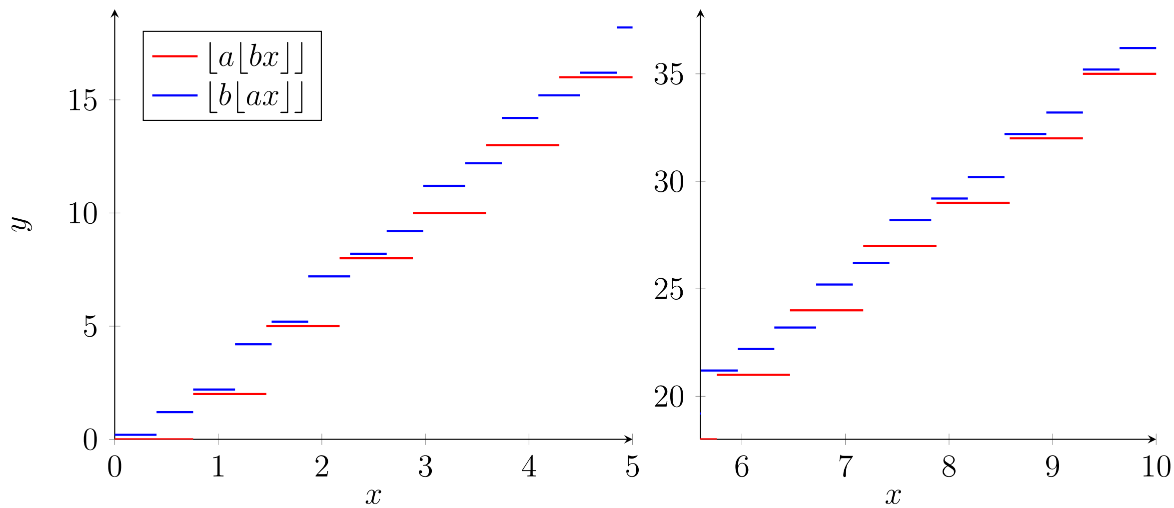

Figure 1: Graphs of the dilated floor functions under composition with $a = 2.7$ and $b = 1.4$. T intervals can be seen for each function across multiple n. $\lfloor b \lfloor ax \rfloor \rfloor$ has been shifted up by 0.2 to better visualize where T intervals overlap.

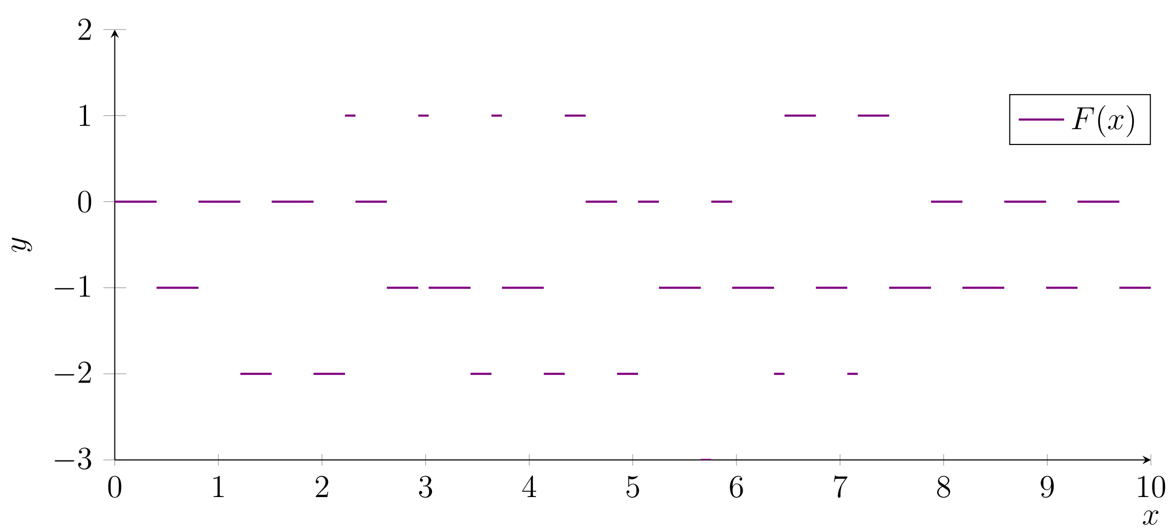

Figure 2: Plot of the difference between dilated floor functions with $a = 2.7$ and $b = 1.4$.

Naturally, the regions of the T intervals that overlap for a particular n will correspond to the zeros of $F(x)$ and, subsequently, the intervals that make up $Z(a, b)$. Notice from Figure 1 that not all n produce a zero interval. Additionally, the functions bounding the zero intervals change, which further justifies the lack of simplification for $Z(a, b)$ when

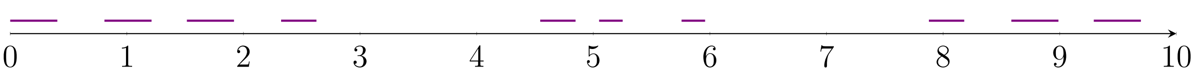

Figure 3: Visualized intervals of $Z(a,b)$, also the zeros of $F(x)$, with $a = 2.7$ and $b = 1.4$.

$ab > 0$. However, if $F(x)$ were periodic with period T, we could simplify the set over which we take the union. Instead of taking the union over the infinite set $n \in \mathbb{Z}$, we can take the union for n in a finite interval. This would ensure we find all the zeros within a full period, and the rest will follow from shifting the intervals by the period.

2 Integer a,b

In this section we consider the special case $a, b \in \mathbb{Z}$, which makes many of our questions easier to answer. Notably, the outer floor functions of $F(x)$ drop away, leaving

$$F(x) = a\lfloor bx \rfloor - b\lfloor ax \rfloor$$

We will characterize $Z(a,b)$ (with a much more manageable expression than Theorem 1.1), calculate the commuting proportion, and define tight bounds on $F(x)$.

Theorem 2.1 *Let $a, b \in \mathbb{Z}$ with $ab \neq 0$. Set $r = \gcd(a,b)$ and set $c = \min(|a|^{-1}, |b|^{-1})$. Then*

$$Z(a,b) = \begin{cases} \bigcup_{n\in\mathbb{Z}} \left\{\frac{n}{r}\right\} & ab < 0 \\ \bigcup_{n\in\mathbb{Z}} \left[\frac{n}{r}, \frac{n}{r} + c\right) & a > 0, b > 0 \\ \bigcup_{n\in\mathbb{Z}} \left(\frac{n}{r} - c, \frac{n}{r}\right] & a < 0, b < 0 \end{cases}$$

We now prove some lemmas that will be useful in the proof of Theorem 2.1. For the remainder of the section, let $r = \gcd(a,b)$.

Lemma 2.2 *Let $a, b \in \mathbb{Z}$, then $F_{a,b}(x)$ is periodic with period $\frac{1}{r}$.*

Proof. $F_{a,b}(x + \frac{1}{r}) = a\lfloor bx + \frac{b}{r} \rfloor - b\lfloor ax + \frac{a}{r} \rfloor$. Since $\frac{a}{r}, \frac{b}{r} \in \mathbb{Z}$, we can pull them out of the floor function. This gives $a\lfloor bx \rfloor + \frac{ab}{r} - b\lfloor ax \rfloor - \frac{ab}{r} = a\lfloor bx \rfloor - b\lfloor ax \rfloor = F_{a,b}(x)$. □

Lemma 2.3 *Let $x, y \in \mathbb{N}$. If $y \nmid x$, then $\lceil \frac{x}{y} \rceil = \lceil \frac{x+1}{y} \rceil$*

Proof. Suppose that $y \nmid x$. Applying the division algorithm on x,y, we have unique $q, s \in \mathbb{Z}$ such that $x = yq + s$, $0 < s < y$. Then $\lceil \frac{x}{y} \rceil = \lceil q + \frac{s}{y} \rceil = q + 1$. Next, $x + 1 = yq + (s + 1)$. Since $s + 1 \leq y$, we have $\lceil \frac{x+1}{y} \rceil = q + 1$. □

Lemma 2.4 *Let $a, b \in \mathbb{Z}$, then $F_{a,b}(x) = -F_{-a,-b}(-x)$*

Proof. $-F_{-a,-b}(-x) = -(-a\lfloor bx\rfloor - (-b)\lfloor ax\rfloor) = a\lfloor bx\rfloor - b\lfloor ax\rfloor = F_{a,b}(x)$ □

Now that we have these lemmas, we can continue with the proof of the above theorem.

Proof of Theorem 2.1. We have three cases: a and b have opposite signs, a and b positive, and a and b negative. In each case, we will find the $x \in Z(a,b)$ for a particular interval of length $\frac{1}{r}$. Then by Lemma 2.2, adding every integer multiple of $\frac{1}{r}$ to each of those x will provide the complete set $Z(a,b)$.
Case 1 ($ab < 0$): Without loss of generality, suppose that $a < 0$ and $b > 0$. By definition, $\lfloor ax\rfloor \leq ax$ and $\lfloor bx\rfloor \leq bx$, so $a\lfloor bx\rfloor \geq abx$ and $b\lfloor ax\rfloor \leq abx$. If $a\lfloor bx\rfloor = b\lfloor ax\rfloor$, then they both must be equal to abx. Then we must have that $\lfloor ax\rfloor = ax$ and $\lfloor bx\rfloor = bx$, which means that $ax, bx \in \mathbb{Z}$. Looking at the interval $\left[0, \frac{1}{r}\right)$, the only x that satisfies this requirement is $x = 0$.
Case 2 ($a > 0$, $b > 0$): We determine which intervals in Theorem 1.1 are nonempty for $n \in \left[0, \frac{ab}{r}\right)$, which corresponds to $x \in \left[0, \frac{1}{r}\right)$. If $n = 0$, our interval is $[0, \frac{1}{b}) \cap [0, \frac{1}{a})$, which is equivalent to $\left[0, \min\left(\frac{1}{a}, \frac{1}{b}\right)\right)$ or $[0, c)$. For the remaining n, since $a \nmid n$ we apply Lemma 2.3 to conclude that the interval $\left[\frac{1}{b}\lceil\frac{n}{a}\rceil, \frac{1}{b}\lceil\frac{n+1}{a}\rceil\right)$ is empty, for any $n \in \left(0, \frac{ab}{r}\right)$.
Case 3 ($a < 0$, $b < 0$): Applying Lemma 2.4, we know the elements in $Z(a,b)$ are the opposite of the elements in $Z(-a,-b)$. Since $-a, -b > 0$, we can use the previous case to conclude that $F_{a,b}(x) = 0$ for $0 \leq -x < c$. Multiplying by -1, the interval becomes $-c < x \leq 0$. This means $(-c, 0] \subset Z(a,b)$ for $x \in (-\frac{1}{r}, 0]$. □

Note that Theorem 2.1 says that $\frac{1}{r}$ is the smallest period of $F_{a,b}$ because the zeros of $F_{a,b}$ in $I = (-\frac{1}{r}, 0]$ if $a, b < 0$, or in $I = [0, \frac{1}{r})$ otherwise, are a nonempty, connected, proper subset of I.

The expressions from Theorem 2.1 allow us to easily find the commuting proportion for integer a, b. Recall that we define commuting proportion $P(a,b)$ as:

$$P(a,b) = \lim_{N\to\infty} \frac{\mu(Z(a,b) \cap [-N,N])}{2N}$$

Given the periodicity of F, we can omit the limit entirely and simply find the proportion over one period. So

$$P(a,b) = r \cdot \mu(Z(a,b) \cap [0, \tfrac{1}{r}))$$

Additionally, $Z(a,b) = Z(b,a)$ and hence $P(a,b) = P(b,a)$.

Theorem 2.5 *Let $a, b \in \mathbb{Z}$ and $c = \min(|a|^{-1}, |b|^{-1})$, then*

$$P(a,b) = \begin{cases} cr & ab > 0 \\ 0 & ab < 0 \end{cases}$$

Proof. This follows directly from the above and Theorem 2.1. □

Corollary 2.6 *For $a, b \in \mathbb{Z}$, if $\frac{b}{a} = n \in \mathbb{N}$, then $P(a,b) = \frac{1}{n}$.*

Proof. Rearranging the expression for n, we have $b = an$. By Theorem 2.5, the commuting proportion is cr. Substituting we get $\frac{\gcd(a,an)}{\max(|a|,|an|)} = \frac{a}{an} = \frac{1}{n}$. □

Corollary 2.7 *Let $a, b, n \in \mathbb{Z}$ with $n \neq 0$. Then $P(na, nb) = P(a, b)$*

Proof. This is a consequence of Theorem 2.5 since n can be factored out of r and $\frac{1}{n}$ can be factored out of c. □

The final question to answer for integer a, b is how extreme $F(x)$ can get. Something that isn't immediately apparent from its definition is the fact that $F(x)$ is bounded both above and below. Since the function measures how far away the composed floor functions are from commuting, this tells us that there is an absolute "worst case" for how far away from commuting $a\lfloor bx\rfloor$ and $b\lfloor ax\rfloor$ are. To help in the proof of this, we first prove an interesting property of the image of $F(x)$.

Lemma 2.8 *Let $a, b \in \mathbb{Z}$, then $F(x) \equiv 0 \pmod r$.*

Proof. Clearly $r \mid a$ and $r \mid b$, so it follows that $r \mid (a\lfloor bx\rfloor - b\lfloor ax\rfloor)$, so $F(x) \equiv 0 \pmod r$ as desired. □

We will now prove that $F(x)$ is bounded, thereby completely answering our original questions for integer a, b.

Theorem 2.9 *Let $a, b \in \mathbb{Z}$, and set $r = \gcd(a, b)$, then $F(x)$ is bounded as follows:*

$$\begin{aligned} r - a \leq &F(x) \leq b - r && a > 0, b > 0 \\ b + r \leq &F(x) \leq -a - r && a < 0, b < 0 \\ r - a + b \leq &F(x) \leq 0 && a > 0, b < 0 \\ 0 \leq &F(x) \leq b - a - r && a < 0, b > 0 \end{aligned}$$

Furthermore, these bounds are tight.

First, we will demonstrate that these bounds are achieved, and then that the rest are in fact bounds. There are eight bounds to check but the calculations are very similar so we shall only check two. The remaining values are summarized in the Figure 4. Recall that the fractional part of x is denoted by $\{x\}$ and equals $x - \lfloor x\rfloor$.

Lemma 2.10 *If $a, b > 0$, then $F(x) = b - r$ when $x = \frac{n}{b}$ and $na \equiv b - r \pmod b$.*

Proof. Plugging in x we have $F(x) = a\lfloor b\frac{n}{b}\rfloor - b\lfloor a\frac{n}{b}\rfloor = a\lfloor n\rfloor - b\lfloor \frac{an}{b}\rfloor = an - b\lfloor\frac{an}{b}\rfloor$. Since $\lfloor\frac{an}{b}\rfloor = \frac{an}{b} - \left\{\frac{an}{b}\right\}$, where $\{x\}$ denotes the fractional part of x, we have $F(x) = an - b(\frac{an}{b} - \left\{\frac{an}{b}\right\}) = b\left\{\frac{an}{b}\right\}$. Since $an \equiv b - r \pmod b$, $\left\{\frac{an}{b}\right\} = \left\{\frac{b-r}{b}\right\}$, and since $0 \leq \frac{b-r}{b} < 1$, $\left\{\frac{b-r}{b}\right\} = \frac{b-r}{b}$. Hence, $F(x) = b\left\{\frac{an}{b}\right\} = b\frac{b-r}{b} = b - r$. □

Lemma 2.11 *If $a, b > 0$, then $F(x) = r - a$ when $x = \frac{n}{a}$ and $nb \equiv a - r \pmod a$.*

Proof. By the same process as above, $F(x) = a\lfloor \frac{bn}{a} \rfloor - b\lfloor \frac{an}{a} \rfloor = a\lfloor \frac{bn}{a} \rfloor - bn = a\left(\frac{bn}{a} - \left\{\frac{bn}{a}\right\}\right) - bn = -a\left\{\frac{bn}{a}\right\} = -a\left\{\frac{a-r}{a}\right\} = -a\frac{(a-r)}{a} = r - a$ □

Listed below are values for x such that the remaining bounds are obtained.[2]

a	b	x	n	$F(x)$	Extreme
>0	>0	n/b	$na \equiv -r \pmod b$	$b-r$	max
>0	>0	n/a	$nb \equiv -r \pmod a$	$r-a$	min
<0	<0	n/a	$nb \equiv r \pmod a$	$-a-r$	max
<0	<0	n/b	$na \equiv r \pmod b$	$b+r$	min
>0	<0	0		0	max
>0	<0	$n/b+\varepsilon$	$na \equiv r \pmod b$	$r-a+b$	min
<0	>0	$n/a+\varepsilon$	$nb \equiv r \pmod a$	$b-a-r$	max
<0	>0	0		0	min

Figure 4: Sufficient conditions for $F(x)$ to attain its extreme values

Now we will continue with the proof of Theorem 2.9

Proof of Theorem 2.9. In all cases, we have that $bx - 1 < \lfloor bx \rfloor \le bx$ and $ax - 1 < \lfloor ax \rfloor \le ax$ by definition. Our different bounds arise from how these inequalities change when we multiply by a and $-b$ respectively. To illustrate this, we explicitly show the first case.

Suppose $a, b > 0$. When we multiply the two inequalities through by a and $-b$, we get $abx - a < a\lfloor bx \rfloor \le abx$ and $-abx \le -b\lfloor ax \rfloor < -abx + b$. Next, we can find bounds for $F(x)$ by adding our two inequalities together. This gives $-a < a\lfloor bx \rfloor - b\lfloor ax \rfloor = F(x) < b$. The rest of the cases will follow by the same process. Taking careful consideration of the signs of a and b when we multiply, we obtain the following information:

$$\begin{aligned} -a <&F(x) < b && a > 0, b > 0 \\ b <&F(x) < -a && a < 0, b < 0 \\ b - a <&F(x) \le 0 && a > 0, b < 0 \\ 0 \le&F(x) < b - a && a < 0, b > 0 \end{aligned}$$

We can now tighten the strict inequalities by utilizing the codomain described by Lemma 2.8. Since any integer combination of a and b is equivalent to 0 mod r, we can conclude that the highest (or lowest) possible value $F(x)$ can take on is the upper bound minus r or the lower bound plus r, where $r = \gcd(a, b)$. The bounds given in the theorem statement follow from this fact. □

[2] Six of the eight extreme values are achieved on a closed interval. However, in the cases where $ab < 0$, the endpoints of the intervals for which the bounds are achieved are both open. Therefore, adding a sufficiently small ε to the left endpoint produces a point inside the interval.

Corollary 2.12 *For real y, define $(y)^+ = \max(y, 0)$.*
Then for $ab > 0$, we have $r - (a)^+ - (-b)^+ \leq F(x) \leq -r + (-a)^+ + (b)^+$.
For $ab < 0$, we have $-(-r + a - b)^+ \leq F(x) \leq (-r - a + b)^+$

Proof. This follows from Theorem 2.9 □

3 Non-integer a,b

In this section we provide some analogous results that are extended to real numbers.

Theorem 3.1 *Let a, b be arbitrary real numbers, then $F_{a,b}(x)$ is bounded as follows:*

$$\begin{aligned} \lfloor -a \rfloor \leq &F_{a,b}(x) \leq \lceil b \rceil && a > 0, b > 0 \\ \lfloor b \rfloor \leq &F_{a,b}(x) \leq \lceil -a \rceil && a < 0, b < 0 \\ \lfloor b - a \rfloor \leq &F_{a,b}(x) \leq 0 && a > 0, b < 0 \\ 0 \leq &F_{a,b}(x) \leq \lceil b - a \rceil && a < 0, b > 0 \end{aligned}$$

Proof. Recall that restricting a and b to be integers let us remove the outer floor functions. We can still follow the same proof structure as Theorem 2.9, but this time, we will apply another floor function and adjust accordingly.
We begin with the definition of the floor function to obtain $bx - 1 < \lfloor bx \rfloor \leq bx$ and $ax-1 < \lfloor ax \rfloor \leq ax$. For all cases, we will multiply these inequalities by a or b, respectively, and then apply another floor function. Next, we multiply the second inequality by -1; finally, combine the inequalities to construct bounds for $F_{a,b}(x)$. We now demonstrate this for the positive case.
Suppose $a, b > 0$. Multiplying through the inequalities and then applying another floor function yields $abx - a - 1 < \lfloor a \lfloor bx \rfloor \rfloor \leq abx$ and $abx - b - 1 < \lfloor b \lfloor ax \rfloor \rfloor \leq abx$. Multiplying the second inequality by -1 and adding our two inequalities together results in $-a - 1 < \lfloor a \lfloor bx \rfloor \rfloor - \lfloor b \lfloor ax \rfloor \rfloor = F_{a,b}(x) < b + 1$.
Doing this for the rest of the cases provides this information:

$$\begin{aligned} -a - 1 < &F_{a,b}(x) < b + 1 && a > 0, b > 0 \\ b - 1 < &F_{a,b}(x) < -1 - a && a < 0, b < 0 \\ b - a - 1 < &F_{a,b}(x) < 1 && a > 0, b < 0 \\ -1 < &F_{a,b}(x) < b - a + 1 && a < 0, b > 0 \end{aligned}$$

Since the codomain of F is $\mathbb{Z}$, we can tighten these bounds by subtracting 1 from the upper bounds and taking its ceiling, and adding 1 to the lower bounds an taking its floor. This brings the bounds to those in the statement. □

This is an analog of Theorem 2.9. However, the bounds are not necessarily known to be tight.

Based on the integer case in the previous section, one may wonder if Theorem 1.1 can be written more concisely for real a, b. However, we will show that $F(x)$ isn't always periodic. This serves as justification for taking the union over all integer n.

Theorem 3.2 *For any $a > 1$ and $b \in (-1, 0)$ such that $\frac{a}{b} \notin \mathbb{Q}$, $F(x)$ is not periodic.*

Proof. We will prove this by contradiction. Assume $F(x)$ is periodic, and let $a > 1$, $b \in (-1, 0)$, and $\frac{a}{b} \notin \mathbb{Q}$. Theorem 1.1 says all the zeros of $F(x)$ are given by $\bigcup_{n\in\mathbb{Z}} \left[\frac{1}{a}(\lfloor\frac{n+1}{b}\rfloor + 1), \frac{1}{b}\lceil\frac{n}{a}\rceil\right]$. Furthermore, the length of the segments of the zeros are given by the expression $\frac{1}{b}\lceil\frac{n}{a}\rceil - \frac{1}{a}(\lfloor\frac{n+1}{b}\rfloor + 1)$. When $n = 0$, the length is $0 - \frac{1}{a}(\lfloor\frac{1}{b}\rfloor + 1)$. Notice that for our particular choice of a, b, this value is positive, and therefore the zero interval is nonempty. Because of our assumption that the function is periodic, there must be an infinite number of zero intervals of the same length. In other words, there must be an infinite number of $n \in \mathbb{Z}$ that satisfy $\frac{1}{b}\lceil\frac{n}{a}\rceil - \frac{1}{a}(\lfloor\frac{n+1}{b}\rfloor + 1) = -\frac{1}{a}(\lfloor\frac{1}{b}\rfloor + 1)$. Multiplying by $-a$ and simplifying yields $\frac{-a}{b}\lceil\frac{n}{a}\rceil + \lfloor\frac{n+1}{b}\rfloor = \lfloor\frac{1}{b}\rfloor$. Rearranging slightly gives $\frac{-a}{b}\lceil\frac{n}{a}\rceil = -\lfloor\frac{n+1}{b}\rfloor + \lfloor\frac{1}{b}\rfloor$. Notice the RHS is an integer and that $\frac{-a}{b}$ is irrational.

Examine the cases when $n \geq 1$ and $n \leq -a$. This means $\frac{n}{a} > 0$ for the former and $\frac{n}{a} \leq -1$ for the latter. In both situations, $|\lceil\frac{n}{a}\rceil| \geq 1$. It follows that $\frac{-a}{b}\lceil\frac{n}{a}\rceil$ will be irrational. Because the LHS of the equality is irrational while the RHS is an integer, no values of n within these intervals satisfy the equality.

This means all solutions to the equation lie within the interval $-a < n \leq 0$. However, this is a finite interval for $n \in \mathbb{Z}$. It then follows that there are only a finite number of solutions, which corresponds to a finite number of zero intervals of that particular length. However, this is a contradiction, as there must be an infinite number, and our assumption that $F(x)$ is periodic for these particular a, b is false. Therefore $F(x)$ is not periodic. $\square$

We determined an expression for the commuting proportion $P(a, b)$ in the special case when $a, b \in \mathbb{Z}$, but we were also interested in a similar result for general a, b. In the case where a, b are not integers, we do not have the simplified intervals for $Z(a, b)$ of Theorem 2.1, but instead must work with the intervals of Theorem 1.1. It turns out for rational a, b, $F(x)$ is periodic, and we can leverage that fact to again remove the limit portion of the $P(a, b)$ definition, which allows us to calculate (slowly) the exact commuting proportion for rational a, b. We will now find a period of $F(x)$.

Lemma 3.3 *Let $a = \frac{n_1}{d_1}$, $b = \frac{n_2}{d_2}$, where $n_1, n_2, d_1, d_2 \in \mathbb{Z}$ with $d_1, d_2 \neq 0$. Then $F_{a,b}(x)$ is periodic with period $\frac{d_1 d_2}{\gcd(n_1,n_2)}$.*

Proof. From the definition of floor functions, it is clear that if $aT, bT, abT \in \mathbb{Z}$, then T is a period of $F_{a,b}(x)$. Let $T = \frac{d_1 d_2}{\gcd(n_1,n_2)}$. It follows $aT = \frac{n_1 d_2}{\gcd(n_1,n_2)}$. Notice that $\gcd(n_1, n_2)|n_1$, so $aT \in \mathbb{Z}$. In the same way $bT = \frac{d_1 n_2}{\gcd(n_1,n_2)} \in \mathbb{Z}$ and $abT = \frac{n_1 n_2}{\gcd(n_1,n_2)} \in \mathbb{Z}$. Therefore $T = \frac{d_1 d_2}{\gcd(n_1,n_2)}$ is a period of $F_{a,b}(x)$. $\square$

Note that setting $d_1 = d_2 = 1$ immediately gives Lemma 2.2. Using this result, we can compute the exact commuting proportion for a given $a, b \in \mathbb{Q}$, which yields Figure 5. This heat map elicits many conjectures about the commuting proportion. For example, there seem to be many dotted lines of various slopes. From Corollary 2.6, it is clear why the points with integer coordinates that fall on those lines have the same commuting proportion. This raises the question whether analogous results exist for $\mathbb{Q}$. Another

interesting observation is the existence of the parabola-like curves of various lightness symmetric across the line $a = b$.
We also noticed the white and black regions along the axes. It appears like a strong pattern at the integers, with a weaker version at the half integers, and a weaker still at the third-integers and so on.

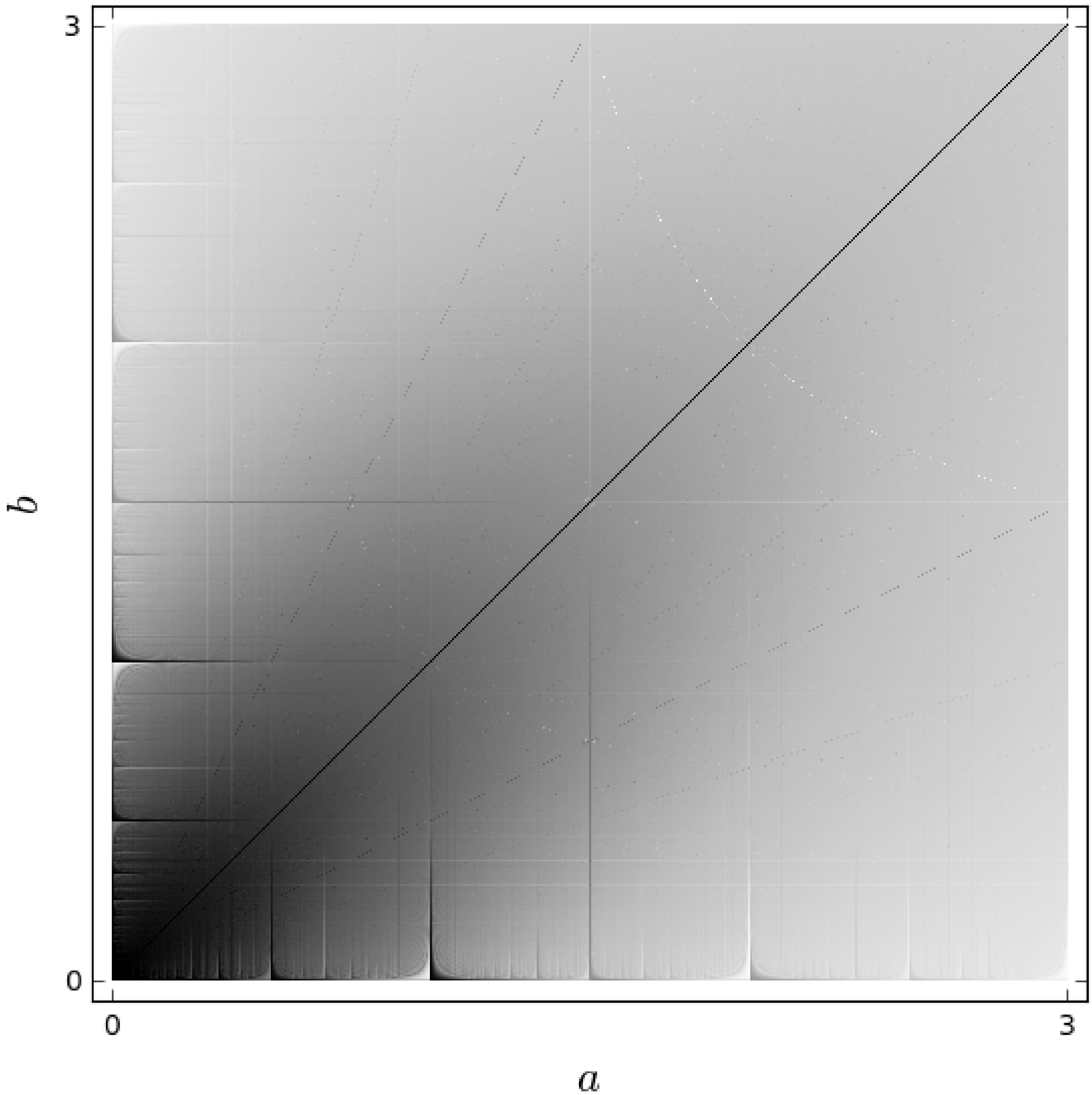

Figure 5: Commuting Proportion from 0 to 3. Each axis contains 1,000 equispaced points (1,000,000 total points), where the each proportion is represented by a grayscale value (the closer to 1, the darker the proportion)

Recall that the greatest common denominator function can be extended to include rational arguments as follows: $\gcd(\frac{n_1}{d_1}, \frac{n_2}{d_2}) = \frac{\gcd(n_1,n_2)}{\operatorname{lcm}(d_1,d_2)}$.

Conjecture 3.4 For arbitrarily chosen $a, b \in \mathbb{Q}^+$, $\exists M$ such that for $m \in \mathbb{Z}$, $m \geq M$, the commuting proportion of $F_{am,bm}(x)$ is given by $\frac{\gcd(a,b)}{\max(a,b)}$.

This is a natural generalization of the integer case. We also conjecture that this such M is a function of both the denominators and numerators of a, b when written as the quotient of relatively prime integers. Furthermore, we believe M is at most

$$\left\lceil \frac{\operatorname{lcm}(d_1, d_2) - \min(d_1, d_2) + 1}{\gcd(n_1, n_2)} \right\rceil.$$

These conjectures appear to hold with computational assistance and when compared to the integer case (with $d_1 = d_2 = 1$).

If true, this conjecture would explain the "lines of similar proportion" that we see in the heat map of commuting proportions. It would also imply that for any line given by $y = \frac{a}{b}x$ on the heat map, all points on the line begin to take on the same value as x becomes large.

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