Algebraic Curvature Tensors of Einstein and Weakly Einstein Model Spaces

R. Shapiro*

Abstract - This research investigates the restrictions on the symmetric bilinear form φ with algebraic curvature tensor $R = R_\varphi$ in Einstein and weakly Einstein model spaces. We show that if a model space is Einstein and has a positive definite inner product, then: if the scalar curvature $\tau \geq 0$, the model space has constant sectional curvature, and if $\tau < 0$, the matrix associated to φ can have at most two eigenvalues. We also show that, given $R = R_\varphi$, a model space is weakly Einstein if and only if R_{φ^2} has constant sectional curvature.

Keywords : Einstein model space; weakly Einstein model space; algebraic curvature tensor

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1 Introduction

An ***algebraic curvature tensor*** R over a finite-dimensional real vector space V is a multilinear function $R : V^{\times 4} \to \mathbb{R}$ satisfying:

$$R(x, y, z, w) = -R(y, x, z, w) = R(z, w, x, y) \text{ and}$$

$$R(x, y, z, w) + R(z, x, y, w) + R(y, z, x, w) = 0,$$

the latter termed the *Bianchi Identity.* Algebraic curvature tensors may be obtained by restricting the Riemannian curvature tensor to a tangent space of a Riemannian manifold. An understandng of algebraic curvature tensors would therefore grant us a subsequent understanding of the ambient manifold.

The following are several preliminary definitions to aid in the understanding of the study of algebraic curvature tensors.

Definition 1.1 *Given vector space V, a **symmetric bilinear form** φ: $V \times V \to \mathbb{R}$ is:*

1. *Symmetric: $\varphi(x, y) = \varphi(y, x)$ for all $x, y \in V$, and*
2. *Linear in the first slot: $\varphi(ax_1 + x_2, y) = a\varphi(x_1, y) + \varphi(x_2, y)$ for all $x_1, x_2, y \in V$.*

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A metric $\langle\cdot,\cdot\rangle$ with respect to basis $\{e_1, e_2, \ldots, e_n\}$ is a symmetric bilinear form that will be notated: $\langle e_i, e_j\rangle = g_{ij}$. For an orthonormal basis, $g_{ij} = \pm\delta_{ij}$. In this paper, we will consider only positive definite metrics, so $g_{ij} = \delta_{ij}$.

We will use a similar notational convention for algebraic curvature tensors: given any basis $\{e_1, e_2, \ldots, e_n\}$, we write $R_{ijkl} = R(e_i, e_j, e_k, e_l)$.

Definition 1.2 *Given vector space V of dimension n, a metric $\langle\cdot,\cdot\rangle$, and an algebraic curvature tensor R, a* ***model space*** *$\mathcal{M}$ is the triple:*

$$\mathcal{M} = (V, \langle\cdot,\cdot\rangle, R).$$

Henceforth, we will use $\mathcal{M}$ interchangably with V due to our particular interest in the space V with an inner product structure and an associated algebraic curvature tensor.

Definition 1.3 *A* ***canonical algebraic curvature tensor*** *R_φ is an algebraic curvature tensor that can be expressed as*

$$R_\varphi(x, y, z, w) = \varphi(x, w)\varphi(y, z) - \varphi(x, z)\varphi(y, w),$$

where φ is a symmetric bilinear form.

We will only consider canonical algebraic curvature tensors in this paper. That is, every $R = R_\varphi$ for some symmetric bilinear form φ.

Given a manifold $\mathfrak{M}$ with smooth metric g and point $P \in \mathfrak{M}$, let $(V_P, \langle\cdot,\cdot\rangle)$ be the tangent inner product space at P. Then, it is possible to construct $\mathcal{M}_P = (V_P, \langle\cdot,\cdot\rangle, R)$, a model space tangent to $\mathfrak{M}$ at P. In this paper, we will explore model spaces that satisfy the *Eisntein* and *weakly Einstein* conditions.

Definition 1.4 *[5] The* ***Ricci tensor*** *ρ over a vector space V of dimension n and orthonormal basis $\{e_1, ...e_n\}$ is defined by:*

$$\rho(x, y) = \sum_{i=1}^{n} R(x, e_i, e_i, y).$$

Definition 1.5 *[5]The* ***scalar curvature*** *τ of model space $\mathcal{M}$ is defined by:*

$$\tau = \sum_{i=1}^{n} \rho(e_i, e_i)$$

on an orthonormal basis $\{e_1, \ldots, e_n\}$ for V.

Note that both the Ricci tensor and scalar curvature are independent of choice of orthonormal basis.

Definition 1.6 *[5] A model space $\mathcal{M}$ is **Einstein** if the Ricci tensor is a scalar multiple of the metric. That is for some $\lambda \in \mathbb{R}$,*

$$\rho(\cdot, \cdot) = \lambda\langle \cdot, \cdot \rangle.$$

*We will call λ the **Einstein constant.** Furthermore, it must be that $\lambda = \frac{\tau}{n}$, where $n = dim(V)$.*

Definition 1.7 *[1] A model space $\mathcal{M}$ is **weakly Einstein** if, given an orthonormal basis $\{e_1, \ldots, e_n\}$ of V:*

$$\sum_{a,b,c=1}^{n} R_{abci}R_{abcj} = \mu\delta_{ij}, \qquad i, j = 1, \ldots, n,$$

*where $\mu = \frac{1}{n}\sum_{w,x,y,z=1}^{n} R_{wxyz}^2$. We will call μ the **weakly Einstein constant**.*

Given that a model space is Einstein or Weakly Einstein with $R = R_\varphi$, we are able to solve for the symmetric bilinear form φ that generates all possible canonical algebraic curvature tensors. In Section 2, we define a matrix Φ associated to the symmetric bilinear form φ. In Section 3, we consider Einstein model spaces, in which φ depends on the dimension of the space. In particular, certain Einstein model spaces have a property called *constant sectional curvature*, introduced in Definition 3.3. These results are summarized by the Main Theorems for Einstein Model Spaces:

Main Theorem for Einstein Model Spaces 1 $\tau \geq 0$. *Given $R = R_\varphi$, if $\mathcal{M}$ is Einstein with scalar curvature $\tau \geq 0$ and $dim(\mathcal{M}) = n$, $\mathcal{M}$ has constant sectional curvature $\frac{\tau}{n(n-1)}$.*

Main Theorem for Einstein Model Spaces 2 $\tau < 0$. *Let $\mathcal{M}$ be an Einstein model space of dimension n with $R = R_\varphi$ and scalar curvature $\tau < 0$, and suppose the eigenvalues of the matrix assocated to φ are $\lambda_1, \ldots, \lambda_n$. Then, there are at most two distinct eigenvalues, x and y, with opposite signs. Let j be the multiplicity of x and k the multiplicity of y. Supposing without loss of generality that $x > 0$ and $y < 0$, we find*

$$x = \sqrt{\frac{n|\tau|(k-1)}{j-1}} \text{ and } y = -\sqrt{\frac{n|\tau|(j-1)}{k-1}}.$$

In Section 4, we explore weakly Einstein model spaces, in which φ is determined by signs of the eigenvalues of Φ. This relationship is presented in the Main Theorem for Weakly Einstein Model Spaces:

Main Theorem for Weakly Einstein Model Spaces 1 *Let φ^2 be the symmetric bilinear form with associated linear operator Φ^2. Let $\mathcal{M}$ be a model space with $R = R_\varphi$. Let $\tilde{\mathcal{M}}$ be a model space with $R = R_{\varphi^2}$ and the same metric as $\mathcal{M}$. Then, $\mathcal{M}$ is weakly Einstein if and only if $\tilde{\mathcal{M}}$ has constant sectional curvature.*

These theorems have an important corollary:

Main Corollary 1 *In model spaces, the Einstein condition does not imply the weakly Einstein condition for dim($\mathcal{M}$) ≥ 5.*

2 Diagonalization and Eigenvalues of Φ

To facilitate the discussion of canonical algebraic curvature tensors, we define a matrix Φ for each symmetric bilinear form φ. Let φ be a symmetric bilinear form. Given a model space $\mathcal{M}$ and an orthonormal basis $\{e_1, e_2, \ldots, e_n\}$ of $\mathcal{M}$, it is possible to express φ as the matrix:

$$\varphi = \begin{pmatrix} \varphi(e_1, e_1) & \varphi(e_1, e_2) & \ldots & \varphi(e_1, e_n) \\ \varphi(e_2, e_1) & \varphi(e_2, e_2) & \ldots & \varphi(e_2, e_n) \\ \vdots & \vdots & \ddots & \vdots \\ \varphi(e_n, e_1) & \varphi(e_n, e_2) & \ldots & \varphi(e_n, e_n) \end{pmatrix}.$$

There also exists a unique associated operator $\Phi : V \to V$ defined by:

$$\varphi(x, y) = \langle \Phi x, y \rangle.$$

Furthermore, Φ is self-adjoint due to the symmetry of φ. If Φ^* is the adjoint of Φ with respect to inner product $\langle \cdot, \cdot \rangle$, then:

$$\langle \Phi x, y \rangle = \varphi(x, y) = \varphi(y, x) = \langle \Phi y, x \rangle = \langle y, \Phi^* x \rangle = \langle \Phi^* x, y \rangle.$$

Thus, $\Phi = \Phi^*$, so Φ is self-adjoint.

In the case that the metric is positive definite, the associated matrix Φ of φ on any basis can be diagonalized on an orthonormal basis [3] $\{e_1, \ldots, e_n\}$, so that $\Phi(e_i) = \lambda_i e_i$, where $\{\lambda_i\}_{1 \leq i \leq n}$ are the eigenvalues of Φ. Thus, this matrix is:

$$\Phi = \begin{pmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \lambda_n \end{pmatrix}.$$

Furthermore, we notice that since $\varphi(e_i, e_j) = \langle \Phi e_i, e_j \rangle = \langle \lambda_i e_i, e_j \rangle = \lambda_i g_{ij} = \lambda_i \delta_{ij}$, the representation of φ as a matrix (above) is equivalent to Φ. So, the matrix representation of φ when Φ is diagonal is:

$$\varphi = \begin{pmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \lambda_n \end{pmatrix}.$$

Thus, if it is given that $R = R_\varphi$ and that the orthonormal basis for V is such that Φ is diagonal, the only possible nonzero entries of R are those given by $R_{ijji} = \lambda_i \lambda_j = -R_{ijij}$

for $i \neq j$. This can be demonstrated by a simple manipulation of the expression for canonical algebraic curvature tensors.

Given these simplifications, we will now explore their implications on Einstein model spaces and solve for the eigenvalues of Φ given $R = R_\varphi$.

3 Einstein Model Spaces

Using facts that become obvious through the simplified notation introduced in Section 2, we may summarize the Einstein condition using the following system of equations:

Proposition 3.1 *Let $\mathcal{M}$ be a model space of dimension at least 2 with $R = R_\varphi$ and a positive definite metric. Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of Φ. Then, the following system of equations holds if and only if $\mathcal{M}$ is Einstein with Einstein constant λ:*

$$\begin{array}{rcl} \lambda & = & \lambda_1(\lambda_2 + \lambda_3 + \ldots + \lambda_n) \\ \lambda & = & \lambda_2(\lambda_1 + \lambda_3 + \ldots + \lambda_n) \\ \lambda & = & \lambda_3(\lambda_1 + \lambda_2 + \ldots + \lambda_n) \\ & \vdots & \\ \lambda & = & \lambda_i(\lambda_1 + \lambda_2 + \ldots + \lambda_{i-1} + \lambda_{i+1} + \ldots + \lambda_n) \\ & \vdots & \\ \lambda & = & \lambda_n(\lambda_1 + \lambda_2 + \ldots + \lambda_{n-1}). \end{array} \tag{1}$$

Proof. Let $\{e_1, \ldots, e_n\}$ an orthonormal basis for V that diagonalizes φ. Recall that, by definition, $\mathcal{M}$ is Einstein when $\rho(e_i, e_j) = \lambda\delta_{ij}$ for all $1 \leq i, j \leq n$. The discussion in Section 2 gives us that when $R = R_\varphi$ and $i \neq j$, $\sum_{k=1}^n R_{ikkj} = 0$. Thus, $\mathcal{M}$ is Einstein if and only if $\rho(e_i, e_i) = \lambda$ for all $1 \leq i \leq n$. By the discussion in Section 2, we have that $\rho(e_i, e_i) = \sum_{k=1}^n R_{ikki} = \sum_{k \neq i} \lambda_i\lambda_k = \lambda$. Therefore, $\mathcal{M}$ is Einstein if and only if the equations in System (1) hold. □

Remark 3.2 Proposition 3.1 has clear applications in $dim(\mathcal{M}) \geq 2$; when $n = 1$, $\lambda = \lambda_1$, the sole eigenvalue of Φ.

In the following subsections, we will apply these equations to Einstein model spaces with the purpose of solving for Φ given various parameters for Einstein constant λ and scalar curvature τ.

3.1 Constant sectional curvature in Einstein model spaces.

Now, we will see why Einstein model spaces with $\tau \geq 0$ have a special property called *constant sectional curvature.*

Definition 3.3 *Given model space $\mathcal{M}$, the* ***sectional curvature*** *κ of a 2-dimensional subspace of $\mathcal{M} = (V, \langle\cdot,\cdot\rangle, R)$ spanned by independent $u, v \in V$ is defined by:*

$$\kappa(u, v) = \frac{R(u, v, v, u)}{\langle v, v\rangle\langle u, u\rangle - \langle u, v\rangle^2}.$$

*$\mathcal{M}$ has **constant sectional curvature** $csc(\varepsilon)$ if κ is independent of the chosen subspace.*

It is easy to show that the sectional curvature of a 2-dimensional subspace is independent of the choice of spanning vectors u, v [4]. In this case, the definition simplifies to:

$$\kappa(u,v) = \frac{R(u,v,v,u)}{R_{\langle\cdot,\cdot\rangle}(u,v,v,u)} = \varepsilon \qquad \text{for all independent } u,v \in V.$$

Now, we will ascertain several basic results which will aid us in the description of Einstein model spaces with $\tau \geq 0$.

Lemma 3.4 *Given a model space $\mathcal{M}$, an algebraic curvature tensor R_φ, and $c \in \mathbb{R}$,*

$$R_{c\varphi} = c^2 R_\varphi.$$

Proof. This result can be obtained through a straightforward application of the definition of R_φ:

$$\begin{aligned} R_{c\varphi}(x,y,z,w) &= c\varphi(x,w)\cdot c\varphi(y,z) - c\varphi(x,z)\cdot c\varphi(y,w) \\ &= c^2(\varphi(x,w)\varphi(cy,z) - \varphi(x,z)\cdot\varphi(y,w)) \\ &= c^2 R_\varphi(x,y,z,w). \end{aligned}$$

□

Lemma 3.5 *If $\varphi = \omega\langle\cdot,\cdot\rangle$ for some $\omega \in \mathbb{R}$ and $R = R_\varphi$, then $\mathcal{M}$ has $csc(\omega^2)$.*

Proof. Let $\varphi = \omega\langle\cdot,\cdot\rangle$ and let u, v span a 2-plane in V. Then,

$$R_\varphi = R_{\omega\langle\cdot,\cdot\rangle} = \omega^2 R_{\langle\cdot,\cdot\rangle}$$

by Lemma 3.4. Calculating the sectional curvature yields:

$$\kappa(u,v) = \frac{R_\varphi(u,v,v,u)}{R_{\langle\cdot,\cdot\rangle}(u,v,v,u)} = \frac{\omega^2 R_{\langle\cdot,\cdot\rangle}(u,v,v,u)}{R_{\langle\cdot,\cdot\rangle}(u,v,v,u)} = \omega^2.$$

Thus, $\mathcal{M}$ has $csc(\omega^2)$. □

With these preliminaries, we are now able to solve for the case $\tau = 0$.

Lemma 3.6 *Let $\mathcal{M}$ be an Einstein model space with orthonormal basis $\{e_1, \ldots, e_n\}$ for V and $\rho(\cdot,\cdot) = \lambda\langle\cdot,\cdot\rangle$. Let $\{\lambda_i\}_{1\leq i\leq n}$ be eigenvalues of diagonalized Φ, as in Section 2. If $R = R_\varphi$, then the following are equivalent:*

1. *$\lambda = 0$ and*

2. *$\lambda_i \neq 0$ for at most one $i \in \{1, \ldots, n\}$.*

In this case, $\mathcal{M}$ has $csc(0)$.

Proof. Suppose $\mathcal{M}$ is Einstein. If $\lambda_i = 0$, then

$$\lambda = \lambda_i \left(\sum_{j=1, j \neq i}^{n} \lambda_j \right) = 0,$$

so (2) implies (1).

Conversely, suppose $\lambda = 0$. Let there be i nonzero eigenvalues of Φ, and, for contradiction, suppose without loss of generality that $\lambda_1, \ldots, \lambda_i \neq 0$ for $i \geq 2$. By System (1), for some $a, b \in \{1, \ldots, i\}$, $a \neq b$,

$$\lambda_a \left(\sum_{c=1, c \neq a}^{n} \lambda_c \right) = 0 \text{ and}$$

$$\lambda_b \left(\sum_{c=1, c \neq b}^{n} \lambda_c \right) = 0.$$

Dividing the first equation through by λ_a and the second by λ_b leads to,

$$\sum_{c=1, c \neq a}^{n} \lambda_c = 0 \text{ and}$$

$$\sum_{c=1, c \neq b}^{n} \lambda_c = 0.$$

Then, subtracting the first equation from the second results in

$$\lambda_a - \lambda_b = 0,$$

and simplifying yields

$$\lambda_a = \lambda_b.$$

Since $a, b \in \{1, 2, \ldots, i\}$ were arbitrary,

$$\lambda_1 = \lambda_2 = \cdots = \lambda_i = \eta \neq 0.$$

Furthermore, when this is substituted back into any of $1, 2, \ldots, i$ of System (1), it becomes clear that

$$\eta \left(\eta(i-1) + \sum_{j=i+1}^{n} \lambda_j \right) = 0.$$

So since $\eta \neq 0$ and $\lambda_j = 0$ for $j > i$,

$$\eta(i-1) + \sum_{j=i+1}^{n} \lambda_j = \eta(i-1) = 0.$$

Thus, either $i = 1$, contradicting our assumption that $i \geq 1$, or or $\eta = 0$, contradicting there being more than one nonzero eigenvalue. As such, (1) implies (2). Now that we have shown this to be true, we will further demonstrate that this implies that $\mathcal{M}$ has constant sectional curvature.

Recall that the only nonzero entries of R_φ are those in the format R_{ijji} or R_{ijij} for $i \neq j$. Also, $R_{ijji} = \varphi(e_i, e_i)\varphi(e_j, e_j) = \Phi_{ii}\Phi_{jj} = \lambda_i\lambda_j$. Since at most one $\lambda_i = 0$ when $\lambda = 0$, $R_{ijji} = 0$ for all i, j, and thus $R_\varphi = 0$ identically. It follows that $\mathcal{M}$ has $csc(0)$. $\square$

Now that we found Einstein model spaces with $\tau = \lambda = 0$ to have $csc(0)$, we will show that Einsein model spaces with $\lambda > 0$ also have constant sectional curvature. When $n = dim(\mathcal{V}) \leq 2$, the model space trivially has constant sectional curvature. Lemma 3.7 will consider the non-trival case of $n \geq 3$.

Lemma 3.7 *Given $R = R_\varphi$, if $\mathcal{M}$ is Einstein with $\lambda > 0$ and $n \geq 3$, then $\Phi = cI$. Furthermore, $c = \pm\sqrt{\frac{\lambda}{n-1}}$, so $\mathcal{M}$ has $csc(\frac{\lambda}{n-1})$.*

Proof. Let $\lambda > 0$ and suppose $\varphi \neq c\langle\cdot,\cdot\rangle$. Then, there exists $i \neq j$ such that $\lambda_i \neq \lambda_j$. Due to the symmetries of System (1), we may assume without loss of generality that $\lambda_1 \neq \lambda_2$. Then, subtracting the first two equations of System (1), we find that:

$$(\lambda_1 - \lambda_2)(\lambda_3 + \lambda_4 + \cdots + \lambda_n) = 0.$$

Since $\lambda_1 \neq \lambda_2$, $\lambda_3 + \lambda_4 + \cdots + \lambda_n = 0$. It is now evident that

$$-\lambda_l = \sum_{k=3,k\neq l}^{n} \lambda_k \qquad \forall l \in \{3, \ldots, n\}.$$

Therefore, substituting that into all remaining equations from System (1) yields

$$\lambda = \lambda_l(\lambda_1 + \lambda_2 + \sum_{k=3,k\neq l}^{n} \lambda_k) = \lambda_l(\lambda_1 + \lambda_2 - \lambda_l) \qquad \forall l \in \{3, \ldots, n\}.$$

Summing both sides of equations over $l = 3, \ldots, n$, we find that:

$$\sum_{l=3}^{n} \lambda = \sum_{l=3}^{n} \lambda_l(\lambda_1 + \lambda_2 - \lambda_l) = \sum_{l=3}^{n} \lambda_l(\lambda_1 + \lambda_2) - \sum_{l=3}^{n} \lambda_l^2, \text{ so}$$

$$(n-2)\lambda = (\lambda_1 + \lambda_2)\left(\sum_{l=3}^{n} \lambda_l\right) - \left(\sum_{l=3}^{n} \lambda_l^2\right) = -\left(\sum_{l=3}^{n} \lambda_l^2\right)$$

since $\sum_{l=3}^{n} \lambda_l = 0$. As such, $\lambda > 0$ and $-\left(\sum_{l=3}^{n} \lambda_l^2\right) \leq 0$, this equation is inconsistent. Therefore, it must be the case that $\varphi = c\langle\cdot,\cdot\rangle$ and $\Phi = cI$.
Since all the eigenvalues of Φ are equal, we can write:

$$\lambda = \lambda_i\left((n-1)\,\lambda_i\right) = (n-1)\lambda_i^2.$$

Therefore, solving for λ_i,

$$\lambda_i = \pm\sqrt{\frac{\lambda}{n-1}}.$$

It follows that $c = \pm\sqrt{\frac{\lambda}{n-1}}$, so, by Lemma 3.5, $\mathcal{M}$ has $csc(\frac{\lambda}{n-1})$. $\square$

Now, we are prepared to compile our results into a single theorem.

Theorem 3.8 *Main Theorem for Einstein Model Spaces. Given $R = R_\varphi$, if $\mathcal{M}$ is Einstein with scalar curvature $\tau \geq 0$, $\mathcal{M}$ has constant sectional curvature $\frac{\tau}{n(n-1)}$.*

Proof. Since $\tau \geq 0$, $\lambda \geq 0$. Let $\tau = 0$. Then, by the results of Lemma 3.6, $\mathcal{M}$ is $csc(0)$. Let $\tau > 0$. Then, by Lemma 3.7, $\mathcal{M}$ has constant sectional curvature. $\square$

In this section, we completely described every Einstein model space with a canonical algebraic curvature tensor and nonnegative scalar curvature. Furthermore, we demonstrated that if $\tau \geq 0$, then the model space has constant sectional curvature.

3.2 Einstein model spaces with negative scalar curvature.

The case in which $\tau, \lambda \geq 0$ in Einstein spaces has already been solved in Section 3.1, leaving the case in which $\tau, \lambda < 0$. To solve for the eigenvalues of Φ, we will first establish that there exist at most 2 distinct eigenvalues of Φ.

Theorem 3.9 *Given an Einstein model space $\mathcal{M}$, if $R = R_\varphi$, then Φ can have at most 2 distinct eigenvalues.*

Proof. If $dim(V) \leq 2$, Φ is smaller than or equal to a 2×2 matrix, and therefore can have at most 2 eigenvalues.

Suppose then that $dim(V) \geq 3$, and Φ has at least 3 distinct eigenvalues. Due to the symmetries of System (1), we can let $\lambda_1 = X$, $\lambda_2 = Y$, and $\lambda_3 = Z$, where X, Y, and Z are distinct nonzero constants. (The case in which any one of $\{X, Y, Z\}$ is zero is covered in Lemma 3.6.) Manipulating the first three equations in System (1) yields:

$$\begin{aligned} \tfrac{\lambda}{X} &= Y + Z + \lambda_4 + \ldots + \lambda_n \\ \tfrac{\lambda}{Y} &= X + Z + \lambda_4 + \ldots + \lambda_n \\ \tfrac{\lambda}{Z} &= X + Y + \lambda_4 + \ldots + \lambda_n \end{aligned}$$

Subtracting the second equation from the first and simplifying shows that

$$\tfrac{\lambda}{X} - \tfrac{\lambda}{Y} = Y - X$$
.

So, $\lambda = XY$. Similar operations for the second and third equations, as well as the first and third equations, lead to the conclusion that $\lambda = XY = YZ = XZ$. Since X, Y, and Z are nonzero, $X = Y = Z$, which contradicts their being distinct. Thus, Φ can have at most 2 distinct eigenvalues. $\square$

Now that we have established that any Φ must have at most two distinct eigenvalues, we will use this information to solve for the two eigenvalues x, y in the case that $\tau, \lambda < 0$. To do this, we will assign j to be the number of times x is an eigenvalue of Φ (multiplicity of x), and k to be the multiplicity of y. We will then use all the information we have gathered thus far to solve for x and y. The results are given in Theorem 3.10.

Theorem 3.10 *Main Theorem for Einstein Model Spaces. Let $\mathcal{M}$ be an Einstein model space with $R = R_\varphi$ and scalar curvature $\tau < 0$, and suppose the eigenvalues of Φ are $\lambda_1, \ldots, \lambda_n$. Then, there are at most two distinct eigenvalues, x and y, with opposite signs. Let j be the multiplicity of x and k the multiplicity of y. Supposing $x > 0$ and $y < 0$, we find that*

$$x = \sqrt{\frac{n|\tau|(k-1)}{j-1}} \text{ and } y = -\sqrt{\frac{n|\tau|(j-1)}{k-1}}. \tag{2}$$

Proof. By Theorem 3.9, $\mathcal{M}$ has at most 2 eigenvalues. Since $\tau < 0$, we know that $\lambda < 0$. Since $\lambda = xy$, as in Theorem 3.9, one of $\{x, y\}$ must be negative and the other positive, and we may assume that $x > 0$ and $y < 0$ without loss of generality. The following system of equations can be compiled from the equations presented in previous sections:

$$\lambda = xy, \tag{3}$$

from the proof of Theorem 3.9,

$$n = j + k, \tag{4}$$

by definition, and

$$(j-1)x + (k-1)y = 0 \tag{5}$$

Equation 5 is derived from System (1), which states that, in this case,

$$\lambda = x(y + (j-1)x + (k-1)y) = xy + x((j-1)x + (k-1)y).$$

Thus, $x((j-1)x + (k-1)y) = 0$, and dividing by x yields: $(j-1)x + (k-1)y = 0$.

Note that $j, k \neq 1$ since that would imply that either x or y equals zero, which contradicts their being nonzero. If j or k is zero, Φ has only one eigenvalue, which simplifies to the $\Phi = cI$ case covered in Section 3.

Manipulating Equation 5 and combining it with Equation 3 yields:

$$\frac{\lambda}{x} = y = -\frac{j-1}{k-1} \cdot x.$$

Since $\lambda < 0$, $-\lambda = |\lambda|$, so $x^2 = |\lambda|\frac{k-1}{j-1}$. Thus,

$$x = \sqrt{|\lambda|\frac{k-1}{j-1}}, \qquad y = -\sqrt{|\lambda|\frac{j-1}{k-1}}.$$

Hence,

$$(\lambda_1, \ldots, \lambda_j, \lambda_{j+1}, \ldots, \lambda_n) =$$
$$\left(\sqrt{|\lambda|\tfrac{k-1}{j-1}}, \ldots, \sqrt{|\lambda|\tfrac{k-1}{j-1}}, -\sqrt{|\lambda|\tfrac{j-1}{k-1}}, \ldots, -\sqrt{|\lambda|\tfrac{j-1}{k-1}}\right).$$

Factoring the right hand side and substituting in $n|\tau|$ for λ leads to the conclusion that:

$$(\lambda_1, \ldots, \lambda_j, \lambda_{j+1}, \ldots, \lambda_n) = \sqrt{n|\tau|}\sqrt{\frac{k-1}{j-1}}\left(1, \ldots, 1, -\frac{j-1}{k-1}, \ldots, -\frac{j-1}{k-1}\right).$$

□

4 Weakly Einstein Model Spaces

Similar to how we described the canonical algebraic curvature tensors of Einstein model spaces in the previous section, we may solve for the algebraic curvature tensors of weakly Einstein model spaces by first constructing a system of equations.

Proposition 4.1 *Let $\mathcal{M}$ be a model space with $R = R_\varphi$ and a positive definite metric. Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of Φ. Then, the following system of equations holds if and only if $\mathcal{M}$ is weakly Einstein:*

$$\begin{array}{rcl}
\tilde{\mu} & = & \lambda_1^2(\lambda_2^2 + \lambda_3^2 + \ldots + \lambda_n^2) \\
\tilde{\mu} & = & \lambda_2^2(\lambda_1^2 + \lambda_3^2 + \ldots + \lambda_n^2) \\
\tilde{\mu} & = & \lambda_3^2(\lambda_1^2 + \lambda_2^2 + \ldots + \lambda_n^2) \\
& \vdots & \\
\tilde{\mu} & = & \lambda_i^2(\lambda_1^2 + \lambda_2^2 + \ldots + \lambda_{i-1}^2 + \lambda_{i+1}^2 + \lambda_n^2) \\
& \vdots & \\
\tilde{\mu} & = & \lambda_n^2(\lambda_1^2 + \lambda_2^2 + \ldots + \lambda_{n-1}^2)
\end{array} \tag{6}$$

where $\tilde{\mu} = \frac{\mu}{2}$.

Proof. Let $\{e_1, \ldots, e_n\}$ an orthonormal basis for V that diagonalizes φ. Recall that $\mathcal{M}$ is weakly Einstein if $\sum_{a,b,c=1}^n R_{abci}R_{abcj} = \mu g_{ij}$. By the discussion in Section 2, $R_{xyyx} = -R_{yxyx}$ are the only nonzero entries of any given R_{abci}, so all other entries may be discarded. Thus, $i = j$ in all nonzero terms. Then,

$$\sum_{a,b,c=1}^n R_{abci}R_{abcj} = \sum_{a=1}^n R_{iaai}^2 + \sum_{a=1}^n R_{aiai}^2 = 2\sum_{a=1}^n R_{iaai}^2 = \mu.$$

Hence,

$$\sum_{a=1}^n R_{iaai}^2 = \frac{\mu}{2} = \tilde{\mu}.$$

Since $R_{iiii} = 0$,

$$\tilde{\mu} = \sum_{a=1, a\neq i}^{n} \lambda_i^2 \lambda_a^2 = \lambda_i^2 \sum_{a=1, a\neq i}^{n} \lambda_a^2 \quad \forall i \in \{1, 2, \ldots, n\}.$$

The logic is reversible, so the converse holds. $\square$

Now that we have established the relationship using a system of equations, we will solve the case for $\mu = 0$.

Lemma 4.2 *Let $\mathcal{M}$ be a weakly Einstein model space with orthonormal basis $\{e_1, \ldots, e_n\}$ of V and μ defined by: $\sum_{a,b,c=1}^{n} R_{abci} R_{abcj} = \mu g_{ij}$. Let $\tilde{\mu} = \frac{\mu}{2}$. Let $\{\lambda_i\}_{1 \leq i \leq n}$ be eigenvalues of Φ, as in Section 2. If $R = R_\varphi$, then the following are equivalent:*

1. *$\mu = 0$ and*
2. *$\lambda_i \neq 0$ for at most one $i \in \{1, \ldots, n\}$.*

In this case, $\mathcal{M}$ has $csc(0)$.

Proof. The proof for this lemma is identical to that of Lemma 3.6, but with λ_i^2 instead of λ_i. $\square$

4.1 Weakly Einstein model spaces with $dim(\mathcal{M}) = n$ and $\mu > 0$.

In this subsection, we will determine that if a model space is weakly Einstein, accociated linear operator Φ has one eigenvalue, up to sign. The following theorem encapsulates this result.

Theorem 4.3 *Main Theorem for Weakly Einstein Model Spaces. Let φ^2 be the symmetric bilinear form with associated linear operator Φ^2. Let $\mathcal{M}$ be a model space with $R = R_\varphi$. Let $\tilde{\mathcal{M}}$ be a model space with $R = R_{\varphi^2}$ and the same metric as $\mathcal{M}$. Then, $\mathcal{M}$ is weakly Einstein if and only if $\tilde{\mathcal{M}}$ has constant sectional curvature.*

Proof. Suppose $\mathcal{M}$ is weakly Einstein. Then, there exists $\tilde{\mu}$ such that System (6) holds. Let $\eta_i = \lambda_i^2$. Then, we know the following to be true:

$$\Phi^2 = \begin{pmatrix} \lambda_1^2 & 0 & \ldots & 0 \\ 0 & \lambda_2^2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \lambda_n^2 \end{pmatrix} = \begin{pmatrix} \eta_1 & 0 & \ldots & 0 \\ 0 & \eta_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \eta_n \end{pmatrix}$$

Furthermore, the following system holds:

$$\begin{array}{rcl} \tilde{\mu} & = & \lambda_1^2(\lambda_2^2 + \lambda_3^2 + \ldots + \lambda_n^2) & = & \eta_1(\eta_2 + \eta_3 + \ldots + \eta_n) \\ \tilde{\mu} & = & \lambda_2^2(\lambda_1^2 + \lambda_3^2 + \ldots + \lambda_n^2) & = & \eta_2(\eta_1 + \eta_3 + \ldots + \eta_n) \\ \tilde{\mu} & = & \lambda_3^2(\lambda_1^2 + \lambda_2^2 + \ldots + \lambda_n^2) & = & \eta_3(\eta_1 + \eta_2 + \ldots + \eta_n) \\ & \vdots & & & \\ \tilde{\mu} & = & \lambda_n^2(\lambda_1^2 + \lambda_2^2 + \ldots + \lambda_{n-1}^2) & = & \eta_n(\eta_1 + \eta_2 + \ldots + \eta_{n-1}). \end{array}$$

This system clearly satisfies the requirements set by System (1), signifying that model space $\tilde{\mathcal{M}}$ is Einstein with $R = R_{\varphi^2}$. Since $\tilde{\mu} > 0$, Theorem 3.7 states that $\mathcal{M}$ must have constant sectional curvature.

Conversely, suppose $\tilde{\mathcal{M}}$ has constant sectional curvature. In [4], Gilkey proves that constant sectional curvature implies $\Phi = cI$, so it must be the case that $\Phi^2 = \eta I$ for some η. Here, it is known that $\eta \geq 0$. Clearly, this fulfills the requirements for $\mathcal{M}$ to be Einstein, as enumerated in System (1). Then, Φ^2 can be expressed as a diagonal matrix with eigenvalues η, while Φ can be written as:

$$\Phi = \begin{pmatrix} \pm\sqrt{\eta} & 0 & \dots & 0 \\ 0 & \pm\sqrt{\eta} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \pm\sqrt{\eta} \end{pmatrix}.$$

Then, the equations in System (6) are fulfilled with $\tilde{\mu} = (n-1)\eta$. Therefore, $\mathcal{M}$ is weakly Einstein. □

Theorem 4.3 has an important implication, summarized by the following corollary:

Corollary 4.4 *Given the same conditions as in Theorem 4.3, the eigenvalues of* Φ *must be the same up to sign.*

In 2010, it was proven that in four dimensions, if a model space is Einstein, it is also weakly Einstein [2]. We can now demonstrate that this is not true in model spaces of higher dimensions. This is a second major conclusion of Theorem 4.3.

Corollary 4.5 *In model spaces, the Einstein condition does not imply the weakly Einstein condition for* $dim(\mathcal{M}) \geq 5$.

Proof. Take the construction of the eigenvalues of Φ to be dictated by $j = 2$, as described in Theorem 3.10.

Then, if $dim(\mathcal{M}) \neq 4$, the eigenvalues are $x = \sqrt{\frac{n|\tau|(k-1)}{j-1}}$ and $y = -\sqrt{\frac{n|\tau|(j-1)}{k-1}}$; since $j \neq k$, $|x| \neq |y|$. Thus, by 4.4, $\mathcal{M}$ is not weakly Einstein.

Then, it is clear that in dimensions other than 4, the eigenvalues are not negatives of each other, as required by Theorem 4.3 for a model space to be weakly Einstein. Thus, a model space with such an $R = R_\varphi$ is Einstein but not weakly Einstein. □

The following is a concrete example of this concept.

Example 4.6 Einstein does not always imply weakly Einstein. Let $\mathcal{M}$ have $R = R_\varphi$ and define Φ to have eigenvalues

$$(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5) = \sqrt{\frac{1}{2}}(1, 1, 1, -2, -2).$$

Clearly, this satisfies being Einstein with $j = 3, k = 2$. However, for the model space to be weakly Einstein, permuting the squares eigenvalues should not change the result.

That is, $\lambda_1^2(\lambda_2^2+\lambda_3^2+\ldots+\lambda_n^2)=\lambda_2^2(\lambda_1^2+\lambda_3^2+\ldots+\lambda_n^2)$. In this example, this condition is clearly not satisfied:

$$\left(\sqrt{\frac{1}{2}}\right)^2\left(\sqrt{\frac{1}{2}}^2+\sqrt{\frac{1}{2}}^2+\left(-2\sqrt{\frac{1}{2}}\right)^2+\left(-2\sqrt{\frac{1}{2}}\right)^2\right)$$

$$\neq\left(-2\sqrt{\frac{1}{2}}\right)^2\left(\sqrt{\frac{1}{2}}^2+\sqrt{\frac{1}{2}}^2+\sqrt{\frac{1}{2}}^2+\left(-2\sqrt{\frac{1}{2}}\right)^2\right)=\frac{\mu}{2},$$

which is the necessary condition for weakly Einstein. Therefore, Einstein does not imply weakly Einstein.

It is easy to produce model spaces of dimension at least 5 that are Einstein but not weakly Einstein using the same process as in Example 4.6 (by choosing, for example $j=2$). These examples prove Corollary 4.5.

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Roberta Shapiro
Rutgers University
110 Frelinghuysen Road
Piscataway, NJ 08854-8019, USA
E-mail: shapirorh@gmail.com

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