Fixed Points of a Generalized Discrete Baker's Transformation

G. Cossel, W. Horn, and W. Sherman*

Abstract - In this note we investigate the fixed point sets for the iterations of a generalization of the two-dimensional discrete baker's transformation. In particular, we will derive explicit formulas for the fixed points, and the number of fixed points. Moreover, we will show that the set of all fixed points is a closed set. This generalizes some of the known results for the classical baker's transformation.

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1 Introduction

The baker's transformation was first introduced by Hopf [5] in 1934 and has since seen applications in various areas, most notably in encryption and image enhancement [1], [2], [3]. The transformation received its name as its physical manifestation models the kneading of dough, a repeated operation of rolling, folding, and turning. We will consider it as a bijection from the two dimensional torus T^2 to itself, where the torus is obtained the usual way by identifying opposing edges of the unit square $[0, 1) \times [0, 1)$. There are several ways of giving an algebraic expression for the baker's transformation, here we write it as a piecewise defined function (see, for example, [5], [10]):

$$B(x, y) = \begin{cases} (2x, y/2), & \text{if } 0 \le x < 1/2 \\ (2x - 1, (y + 1)/2), & \text{if } 1/2 \le x < 1. \end{cases} \tag{1}$$

This equation refers to cutting the unit square into two pieces and compressing by a factor of 1/2, and then stacking them on top of each other. Many aspects of this transformation have been studied. It is often referred to as a deterministic map, whose iterations show some form of chaos. We refer the reader to [4] or other books on Dynamical Systems for these results.

In this paper we will investigate a generalization of this map, where we cut the unit square into m pieces, compress them by a factor $1/m$, and stack them in a particular order. We will concentrate on the sets of fixed points of iterations of this map. In Section

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2 we will define this map and investigate some of its basic properties. We will then derive explicit formulas for the fixed points of iterations of this map and some consequences in Section 3 and Section 4. In Section 5 we will investigate the cardinality of the sets of fixed points and some topological properties of the set of all fixed points.

2 Basic Properties of the Generalized Baker's Transformation

For the remainder of this paper let $m \geq 2$ be a fixed integer. When cutting the unit square into m equal pieces, the definition of the generalized baker's transformation by means of a piecewise defined function becomes extremely cumbersome. One solution to this problem is to use the β-transformation on $[0, 1)$: $T_\beta(x) = \beta x - \lfloor \beta x \rfloor$ (see, for example [6]). If $\beta > 1$ is an integer, T_β maps each sub-interval of the form $[k/\beta, (k+1)/\beta)$ onto the entire interval $[0, 1)$. We will apply this concept to define generalized baker's transformation.

Definition 2.1 *Let σ be a fixed permutation of the set $\{0, 1, \ldots, m-1\}$ and $(x, y) \in [0, 1) \times [0, 1)$. Define*

$$a(x) = \sigma(\lfloor mx \rfloor), \tag{2}$$

and

$$f_1(x) = mx - \lfloor mx \rfloor, \tag{3}$$

and

$$f_2(x, y) = \frac{y}{m} + \frac{a(x)}{m}. \tag{4}$$

The function

$$\mathcal{F}(x, y) = (f_1(x), f_2(x, y)). \tag{5}$$

maps T^2 to itself and will be called **generalized baker's transformation***.*

The function $a(x)$ is a step function obtained by permuting the steps in the graph of $y = \lfloor mx \rfloor$. We will refer to the permutation $(\sigma(0), \sigma(1), \ldots, \sigma(m-1))$ as the stacking order. For $m = 2$ and σ the identity permutation the function $\mathcal{F}$ in Definition 2.1 is identical to the function B in (1). A schematic visualization of $\mathcal{F}$ for $m = 3$ is shown in Figure 1.

Before establishing some basic properties of the function $\mathcal{F}$ we look at a different and useful interpretation of its definition. To do this consider the base m expansions of the coordinates of a point (x, y).

$$x = \sum_{j=1}^{\infty} \frac{x_j}{m^j}, \tag{6}$$

$$y = \sum_{j=1}^{\infty} \frac{y_j}{m^j}, \tag{7}$$

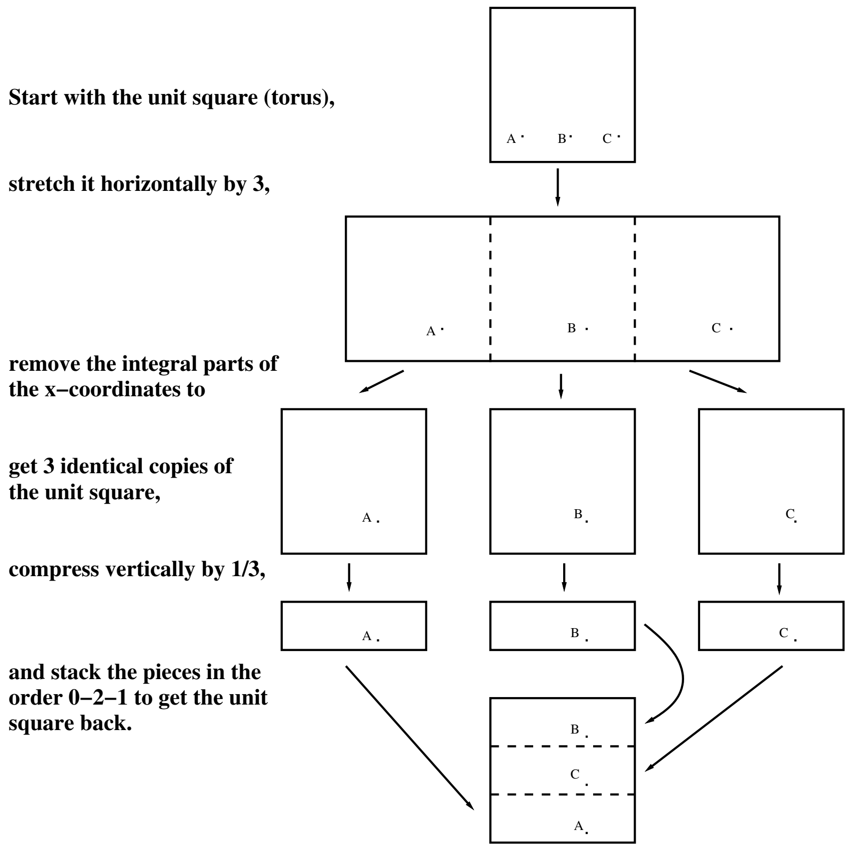

Figure 1: Schematic visualization of $\mathcal{F}$ for $m = 3$ with stacking order $(0, 2, 1)$. Three sample points A, B, and C are also tracked through the map. The first two steps represent the action of f_1, the remaining steps represent the action of f_2. The three sample points are mapped to the same point by f_1, and then mapped to three different points by f_2

where $x_j, y_j \in \{0, 1, \ldots, m-1\}$. It is easy to verify that

$$f_1(x) = \sum_{j=1}^{\infty} \frac{x_{j+1}}{m^j}, \quad \text{and} \quad a(x) = \sigma(x_1). \tag{8}$$

Similarly, we can write

$$f_2(x, y) = \frac{a(x)}{m} + \sum_{j=2}^{\infty} \frac{y_{j-1}}{m^j}. \tag{9}$$

The point which separates the integral and fractional parts of a numbers base m expansion is called the **radix point**. So f_1 acts on the base m expansion of x by moving the radix point to the right by one digit and removing the first digit. f_2 acts on the base m expansion of y by moving the radix point to the left by one digit and replacing the zero created in the first digit to the right of the radix point by $a(x)$. This "sixth grade" intepretation will allow us to make very simple arguments throughout this paper. Base m expansions are not unique, in particular, numbers which have a finite expansion also have a second one that ends with an infinite sequence of the digit $m-1$. In this case we can enforce uniqueness by choosing the finite expansion, i.e. we will require that each expansion satisfies that for each i there exists a $j>i$ such that $x_j \neq m-1$.

We now continue to show some basic properties of $\mathcal{F}$. First, this function is not continuous, for if (x,y) and (x',y) are two points such that

$$x < \frac{k}{m} < x'$$

then the values of $\mathcal{F}(x,y)$ and $\mathcal{F}(x',y)$ will end up in different layers of the image and can be rather far apart, no matter how small $x'-x$ is.

Moreover, the function f_1 is surjective, but not injective, since, if $0 \leq \delta < \frac{1}{m}$ and

$$x_1 = \frac{k}{m} + \delta \quad \text{and} \quad x_2 = \frac{l}{m} + \delta$$

for non-negative integers $k \neq l$, we have that

$$f_1(x_1) = f_1(x_2) = m\delta.$$

On the other hand, the function $\mathcal{F}$ is a bijection from the torus to itself. To show this, it suffices to show that for any $(v,w) \in [0,1) \times [0,1)$ the equation

$$\mathcal{F}(x,y) = (v,w)$$

has a unique solution $(x,y) \in [0,1) \times [0,1)$. To do this we consider first the equation

$$f_2(x,y) = \frac{y}{m} + \frac{a(x)}{m} = w.$$

Solving for y yields

$$y = mw - a(x).$$

Recall that $a(x)$ is integer valued. Since $mw \in [0,m)$ there is a unique integer K, such that $y = mw - K \in [0,1)$. It follows that

$$a(x) = K.$$

Next we consider

$$f_1(x) = mx - \lfloor mx \rfloor = v.$$

This equation has m solutions, given by:

$$x_1 = \frac{v}{m}, \qquad x_2 = \frac{v+1}{m}, \qquad \ldots \qquad x_M = \frac{v+m-1}{m}.$$

These different solutions are located in different sub-intervals of length $1/m$, and therefore, only one of them say x_j will satisfy

$$a(x_j) = K.$$

So the unique solution is given by the pair $(x_j, mw - a(x_j))$.

We now will examine iterates of $\mathcal{F}$. Like previous authors we define an *iteration* of a function f recursively as

$$f^1 = f \quad \text{and} \quad f^{n+1} = f^n \circ f = f \circ f^n,$$

where n is a nonnegative integer, and $\circ$ refers to the usual definition of composition ([8], [9]). In particular, we make the following definition:

Definition 2.2 *Let $\mathcal{F}^j$ denote the j^{th} iteration of $\mathcal{F}$. Define f_1^j and f_2^j by*

$$(f_1^j(x), f_2^j(x, y)) = \mathcal{F}^j(x, y).$$

(It follows from formula (5) that f_1^j is the j^{th} iteration of f_1, but f_2^j is not *the j^{th} iteration of f_2. One can't iterate f_2 because its domain and range have different dimensions.)*

While there are many interesting questions about the behavior of the iterates of this transformation, we restrict our attention to the following questions:

- Given an integer $j \geq 1$, do there exist points (x, y) such that $\mathcal{F}^j(x, y) = (x, y)$? We call these points the *fixed points* of $\mathcal{F}^j$.
- If fixed points exist, is there a closed formula to compute them?
- What is the cardinality of the set of fixed points of $\mathcal{F}^j$?
- What other property does this set of fixed points have?

Except for one case, the map $\mathcal{F}^j$ will always have fixed points. The only example for which $\mathcal{F}^j$ has no fixed points is when $m = 2$, $j = 1$, and σ is the transposition $\sigma(0) = 1$, $\sigma(1) = 0$. In this case

$$\mathcal{F}(x, y) = \begin{cases} (2x - 1, (y + 1)/2) & \text{if } 0 \leq x < 1/2 \\ (2x, y/2) & \text{if } 1/2 \leq x < 1 \end{cases}$$

so there are no fixed points.

In all other cases we will derive explicit formulas for the coordinates of all fixed points of $\mathcal{F}^j$. We also give an explicit formula that answers the second question. Moreover, all fixed points will have rational coordinates.

In the next section we will look exclusively at the first component f_1. We then use this result to study the second component f_2 in Section 4. Whereas the problem of finding fixed points is typically a problem of dynamical systems, our investigation will quickly wander into elementary number theory and the properties of expansions of real numbers in base m.

3 The Fixed Points of f_1^j

Since we are interested in the fixed points of the j^{th} iteration of $\mathcal{F}$, our first aim is to study how the j^{th} iteration affects each component of $\mathcal{F}(x, y)$. If (x^*, y^*) is a fixed point of $\mathcal{F}^j$ then x^* must be a fixed point of f_1^j. So it is natural to consider this one dimensional problem first. We start by giving an explicit formulas for $f_1^j(x)$.

Lemma 3.1 *If the base m expansion of x is given by*

$$x = \sum_{i=1}^{\infty} \frac{x_i}{m^i},$$

then

$$f_1^j(x) = \sum_{i=1}^{\infty} \frac{x_{i+j}}{m^i}. \tag{10}$$

Alternatively, we also have that

$$f_1^j(x) = m^j x - \lfloor m^j x \rfloor. \tag{11}$$

Proof. (10) holds for $j = 1$ by (8). Assume it holds also for $j = k$. Then

$$f_1^{k+1}(x) = f_1(\sum_{i=1}^{\infty} \frac{x_{i+k}}{m^i}) = \sum_{i=1}^{\infty} \frac{x_{i+k+1}}{m^i},$$

and the result follows by induction. As for (11), this formula describes moving the radix point in the base m expansion of x to the right by j positions and removing the integral part of the expansion. $\square$

Before continuing we introduce a new notation for base m expansions. Let x have the expansion

$$x = \sum_{i=1}^{\infty} \frac{x_i}{m^i}.$$

We call this expansion *periodic with period k* if

$$x_i = x_{i+k}$$

for all i. We can now describe the fixed points of f_1^j.

Proposition 3.2 *The following statements are equivalent:*

1. *x^* is a fixed point of f_1^j.*

2. *$x^* \in [0, 1)$ is a rational number with a periodic base m expansion with a period that divides j.*

3. *$x^* = \frac{k}{m^j - 1}$ for some $k \in \{0, 1, \ldots m^j - 2\}$.*

Proof. We will show that (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (1).

(1) $\Rightarrow$ (2): Let x^* be a fixed point of f_1^j with base m expansion

$$x^* = \sum_{j=1}^{\infty} \frac{x_i}{m^i},$$

using (10) we have

$$x^* = \sum_{j=1}^{\infty} \frac{x_i}{m^i} = f_1^j(x^*) = \sum_{i=1}^{\infty} \frac{x_{i+j}}{m^i},$$

and therefore $x_i = x_{i+j}$ for all i. Thus the expansion is periodic with a period equal to j or equal to a factor of j. Since numbers that have periodic expansions are rational, the proof for this step is complete.

(2) $\Rightarrow$ (3): Since $x^* < 1$ has a periodic expansion with a period that divides j we can find an integer $k \in \{0, 1, \ldots, m^j - 2\}$ such that

$$x^* = \sum_{i=1}^{\infty} k \left(\frac{1}{m^j}\right)^i = \frac{k}{m^j} \sum_{i=0}^{\infty} \left(\frac{1}{m^j}\right)^i = \frac{k}{m^j} \frac{1}{1 - \frac{1}{m^j}} = \frac{k}{m^j - 1}.$$

(3) $\Rightarrow$ (1): For this step we will use (11). Observe that

$$m^j \frac{k}{m^j - 1} = k \sum_{i=0}^{\infty} \left(\frac{1}{m^j}\right)^i = k + \frac{k}{m^j} \sum_{i=0}^{\infty} \left(\frac{1}{m^j}\right)^i = k + \frac{k}{m^j - 1}.$$

Next observe that

$$\lfloor k + \frac{k}{m^j - 1} \rfloor = k.$$

Combining these two observatios in the formula 11 yields

$$f_1^j(x^*) = x^*,$$

and the proposition is proven.

$\square$

This proposition has some immediate consequences, first we have:

Corollary 3.3 *The map f_1^j has exactly $m^j - 1$ fixed points, and if x^* is a fixed point of this map so are*

$$f_1(x^*), f_1^2(x^*), \ldots, f_1^{j-1}(x^*).$$

Proof. The first assertion is an immediate consequence of the proposition. For the second observe that

$$f_1^j(f_1^k(x^*)) = f_1^k(f_1^j(x^*)) = f_1^k(x^*).$$

$\square$

We note that the points in the sequence

$$f_1(x^*), f_1^2(x^*), \ldots, f_1^{j-1}(x^*),$$

do not need to be distinct. If they are not, then $f_1^d(x^*) = x^*$ for a $d < j$ and $d|j$. Sequences of this type which contain j distinct numbers are called *invariant sets of* f_1^j *or* j*-cycles.*

Finally, we observe that if $x = \frac{p}{q}$ is rational then the base m expansion of x either terminates or becomes periodic after finitely many digits. If it terminates, we have

$$\frac{p}{q} = \sum_{i=1}^{N} \frac{x_i}{m^i} = \frac{\sum_{i=1}^{N} x_i m^{N-i}}{m^N},$$

and therefore $\gcd(q, m) > 1$. We summarize this in:

Corollary 3.4 *If x is rational, then there exists an integer k such that $f_1^k(x)$ is a fixed point of f_1^j for some j.*

Proof. If the base m expansion of x becomes periodic after k digits, then $f_1^k(x)$ has a periodic base m expansion. If the expansion terminates after k digits, then $f_1^k(x) = 0$. $\square$

4 f_2^j and the fixed points of $\mathcal{F}^j$.

We now turn our attention to f_2. We start this section by deriving an explicit formula for f_2^j. We then use this to develop the existence and uniqueness of fixed points of $\mathcal{F}^j(x, y)$.

Lemma 4.1 *$f_2^j(x, y)$ can be written as*

$$f_2^j(x, y) = \frac{y}{m^j} + \frac{\sigma(x_1)}{m^j} + \frac{\sigma(x_2)}{m^{j-1}} + \cdots + \frac{\sigma(x_j)}{m}, \tag{12}$$

where $x_1, x_2, \ldots, x_j$ are the first j digits of the base m expansion of x, and σ is the permutation that defines the stacking order.

Proof. We prove this by induction on j. From (4) we get

$$f_2(x, y) = \frac{y}{m} + \frac{\sigma(x_1)}{m}.$$

Suppose (12) holds for $j = k$. Observe that the leading digit of the expansion of $f_1^k(x)$ is x_{k+1}. We have

$$\begin{aligned} f_2^{k+1}(x, y) &= \frac{f_2^k(x, y)}{m} + \frac{\sigma(x_{k+1})}{m} \\ &= \frac{1}{m}\left(\frac{y}{m^k} + \frac{\sigma(x_1)}{m^k} + \frac{\sigma(x_2)}{m^{k-1}} + \cdots + \frac{\sigma(x_k)}{m}\right) + \frac{\sigma(x_{k+1})}{m} \\ &= \frac{y}{m^{k+1}} + \frac{\sigma(x_1)}{m^{k+1}} + \frac{\sigma(x_2)}{m^k} + \cdots + \frac{\sigma(x_k)}{m^2} + \frac{\sigma(x_{k+1})}{m}, \end{aligned}$$

which is the desired result. $\square$

We remark here, that f_2^j inserts j new digits between the radix point and the first digit of the base m expansion of y. These new digits are prescribed by the first j digits of the base m expansion of x.

We can now give an explicit formula for the fixed points of $\mathcal{F}^j$.

Proposition 4.2 *Let x^* be a fixed point of f_1^j, with base m expansion*

$$x^* = \sum_{i=1}^{\infty} \frac{x_i^*}{m^i}.$$

Then

1. *if $x^* = \sigma^{-1}(m-1)/(m-1)$, then no fixed point of $\mathcal{F}^j$ has an x coordinate equal to x^*.*
2. *if $x^* \neq \sigma^{-1}(m-1)/(m-1)$, then $\mathcal{F}^j$ has a unique fixed point (x^*, y^*) with x-coordinate equal to x^*. In this case the y-coordinate is given by*

$$y^* = \frac{l}{m^j - 1},$$

where

$$l = \sum_{i=1}^{j} \sigma(x_i^*) m^{i-1}.$$

Moreover, y^* has a periodic base m expansion, with a period that divides j.

Proof. Let x^* be a fixed point of f_1^j. The equation

$$y = f_2^j(x^*, y) = \frac{y}{m^j} + \frac{\sigma(x_1^*)}{m^j} + \frac{\sigma(x_2^*)}{m^{j-1}} + \cdots + \frac{\sigma(x_j^*)}{m},$$

is a linear equation which can be solved for y. The solution is given by:

$$y^* = \frac{m^j}{m^j - 1} \left(\frac{\sigma(x_1^*)}{m^j} + \frac{\sigma(x_2^*)}{m^{j-1}} + \cdots + \frac{\sigma(x_j^*)}{m} \right),$$

which immediately yields the explicit formula for y^*. Furthermore, since $y^* < 1$, we have

$$0 \leq l \leq m^j - 2.$$

Thus at least one of the integers $\sigma(x_1^*), \sigma(x_2^*), \ldots, \sigma(x_j^*)$ is less than $m-1$.

Therefore, there is no solution if and only if $\sigma(x_1^*) = \sigma(x_2^*) = \cdots = \sigma(x_j^*) = m-1$. Since the base m expansion of x^* is periodic of period j this implies that there is no solution if and only if

$$x^* = \sigma^{-1}(m-1) \sum_{i=1}^{\infty} \frac{1}{m^i} = \frac{\sigma^{-1}(m-1)}{m-1}.$$

The periodicity of the base m expansion of y follows immediately from the formula for y^*. This completes the proof. $\square$

5 The cardinality of fixed point sets and some further observations

In [9], the authors compute the total number of fixed points of the j^{th} iteration of the classical baker's transformation to be $2^j - 1$. In Proposition 3.2 we give an explicit formula for the fixed points of f_1^j and it follows immediately that there are $m^j - 1$ such fixed points. Together with Proposition 4.2 this yields that $\mathcal{F}^j$ has either $m^j - 1$ or $m^j - 2$ fixed points. We summarize this result in:

Proposition 5.1 *If $\sigma(m-1) = m-1$ the j^{th} iteration $\mathcal{F}^j$ of the baker's transformation has exactly $m^j - 1$ fixed points, otherwise if $\sigma(m-1) \neq m-1$ it has exactly $m^j - 2$ fixed points.*

Proof. If $\sigma(m-1) = (m-1)$ then $\sigma^{-1}(m-1)/(m-1) = 1$ and every fixed point x^* of f_1^j satisfies $x^* \neq \sigma^{-1}(m-1)/(m-1)$ since $x^* < 1$. Therefore Proposition 4.2 implies that $\mathcal{F}^j$ has $m^j - 1$ fixed points. Otherwise, if $\sigma(m-1) \neq m-1$, then $x^* = \sigma^{-1}(m-1)/(m-1)$ is a fixed point of f_1. However, Proposition 4.2 states that no fixed point of $\mathcal{F}$ can have this x-coordinate. Therefore the total number of fixed points is $m^j - 2$. □

A more interesting question is, how many of these fixed points are new to the j^{th} iteration, and are not fixed points of earlier iterations. Clearly, a fixed point of the second iteration will also be a fixed point of the fourth iteration. However, the fourth iteration has $m^4 - 1$ fixed points, whereas the second iteration has only $m^2 - 1$. We call these **new fixed points** of the j^{th} iteration and denote their number by $N(j)$. Clearly, if $d|j$ and (x_0, y_0) is a fixed point of $\mathcal{F}^d$, then it is also a fixed point of $\mathcal{F}^j$. The following known results are easily verified and hold for the fixed point sets of any iterative process. We include it here for completeness

Proposition 5.2 *Let $N(j)$ be the number of new fixed points of $\mathcal{F}^j$. We have:*

1.
$$N(1) = m - \lambda,$$
and

2.
$$m^j - \lambda = \sum_{d|j} N(d).$$
where $\lambda = 1$ if $\sigma(m-1) = m-1$, and $\lambda = 2$ otherwise.

3. *If d is the largest factor of j except j itself, then*
$$N(j) = m^j - m^d.$$

4. *If p is a prime, then*
$$N(p) = m^p - m$$

5. *For all $j > 1$*

$$N(j) = \sum_{d|j} \mu(d) m^{\frac{j}{d}}$$

where $\mu(d)$ is the Moebius function from elementary number theory.

Proof. Most of these assertions are obvious. For the last assertion observe that the Moebius inversion formula (see, for example, [7]) of the statement

$$m^j - \lambda = \sum_{d|j} N(d)$$

gives

$$N(j) = \sum_{d|j} (m^{\frac{j}{d}} - \lambda)\mu(d)$$

and the result immediately follows from the fact that

$$\sum_{d|j} \mu(d) = 0$$

for $j > 1$. $\square$

We now return to the concept of j-cycles introduced earlier. Let (x^*, y^*) be a new fixed point of $\mathcal{F}^j$, then

$$\mathcal{F}(x^*, y^*), \mathcal{F}^2(x^*, y^*), \ldots, \mathcal{F}^{j-1}(x^*, y^*)$$

all are **new** fixed points of $\mathcal{F}^j$. Clearly they are fixed points. To see that they are new fixed points assume that one of them say $\mathcal{F}^k(x^*, y^*)$ is a fixed point of the i^{th} iteration of $\mathcal{F}$ with $i < j$, then

$$\mathcal{F}^{i+k}(x^*, y^*) = \mathcal{F}^k(x^*, y^*).$$

Since $\mathcal{F}$ is a bijection, we can apply its inverse k times to this equation and obtain:

$$\mathcal{F}^i(x^*, y^*) = (x^*, y^*),$$

thus contradicting the statement that (x^*, y^*) was a new fixed point. Moreover, it is easy to see that the different j-cycles are disjoint. Therefore we have:

Corollary 5.3 *The new fixed points of $\mathcal{F}^j$ lie in disjoint j-cycles and $j|N(j)$.*

Since we did not use the specific form of $\mathcal{F}$ to arrive at this it is clear that this last result also applies to the fixed points of iterations of any bijection.

We finish this section by making some observations about the topology of the set of fixed ponits. First, there is the completely obvious observation that the set

$$F_1 = \left\{x : f_1^j(x) = x \quad \text{for some} \quad j \in \mathbb{N}\right\}$$

is dense in $[0,1)$, since for any $x \in [0,1)$ and $\epsilon > 0$ there is a number j such that

$$\frac{1}{m^j - 1} < \epsilon, \qquad \text{and} \qquad \left| \frac{k}{m^j - 1} - x \right| < \epsilon$$

for some k. A similar argument goes for the y-components of fixed points, they are themselves dense in $[0,1)$. However, this does not imply anything about the density of fixed points of $\mathcal{F}^j$ in T^2. In numerical computations these fixed points appear to be dense, however, they are not as we will prove below.

Proposition 5.4 *There exists a point $(x_0, y_0) \in T^2$ which is not a limit point of*

$$F = \left\{ (x,y) : \mathcal{F}^j(x,y) = (x,y) \quad \textit{for some} \quad j \in \mathbb{N} \right\}.$$

The set F is not dense in T^2.

Proof. Fix an integer $j > 0$ and integers $0 \leq k, l < m^j - 1$. Consider the square

$$\left[\frac{k}{m^j}, \frac{k+1}{m^j} \right) \times \left[\frac{l}{m^j}, \frac{l+1}{m^j} \right).$$

If this square contains no fixed point of any iteration of $\mathcal{F}$, choose (x,y) to be the midpoint of this square. Then for any fixed point $(x^*, y^*) \in F$ we have that

$$\sqrt{(x - x^*)^2 + (y - y^*)^2} \geq \sqrt{\frac{1}{4m^{2j}} + \frac{1}{4m^{2j}}} = \frac{1}{\sqrt{2} m^j},$$

and thus (x,y) cannot be a limit point of F.

If this square contains a fixed point $(x^*, y^*) \in F$, observe that k uniquely determines the first j coefficients of the base m expansion of x^*, which in turn determine the first j coefficients of y^* and therefore the number l itself. It follows that all fixed points with x coordinates in the interval $[k/m^j, (k+1)/m^j)$ must have y coordinates in the interval $[l/m^j, (l+1)/m^j)$, and in particular the square

$$\left[\frac{k}{m^j}, \frac{k+1}{m^j} \right) \times \left[\frac{l+1}{m^j}, \frac{l+2}{m^j} \right)$$

is fixed point free and we can find a point (x,y) in this square which is not a limit point of F in the same way as above. □

In Corollary 3.4 we observed that any rational p will be attracted to a fixed point of f_1^j for some j. We will now look on what happens with the y-coordinates of such a point.

Corollary 5.5 *Let $(x,y) \in [0,1) \times [0,1)$ and x be rational and $\epsilon > 0$. Then there exist a number k and a fixed point $(x^*, y^*) \in F$ such that for any $\epsilon > 0$ we have:*

$$f_1^k(x) = x^*, \qquad \left| f_2(x,y)^{k+nj}(x,y) - y^* \right| < \epsilon,$$

for sufficiently large n.

Proof. Let k be the integer such that $f_1^k(x) = x^*$ from Corollary 3.4 and $y' = f_2^k(x^*, y)$. y' may be irrational. Recall the explicit form of f_2^j:

$$f_2^j(x^*, y') = \frac{y'}{m^j} + \frac{\sigma(x_1^*)}{m^j} + \frac{\sigma(x_2^*)}{m^{j-1}} + \cdots + \frac{\sigma(x_j^*)}{m},$$

The division of y' by m^j has the effect of putting j zeros after the radix point in the base m expansion of y' and moving the original expansion to the right by these j digits. The sum then replaces these zeros with a string of j new digits which are infact the first j digits of y^*. Repeating this action n times $f_2^{nj}(x^*, y')$ will have a base m expansion which agrees with the expansion of y^* for the first nj digits. It follows that

$$\left| f_2^{nj} - y^* \right| < \frac{1}{m^{nj-1}},$$

which immediately implies the result. $\square$

We conclude with mentioning some further concepts which are worth investigating. We start with the geometry of the sets of fixed points. In the case of $m = 2$ the set of fixed points for a given iteration is symmetric about the main diagonal of the unit square. In this more general setting this geometry depends strongly on the particular stacking order. It would be interesting to investigate this dependence both numerically and analytically. We suspect that for the standard stacking a_0 defined in the previous section the sets of fixed points show a similar symmetry as in the case $m = 2$. Given below are images of the fixed points for three iterations of $\mathcal{F}$. On the left the standard stacking $(0, 1, 2)$ was used, where as on the right we used the stacking $(0, 2, 1)$. For this particular choice the symmetry seems to still hold.

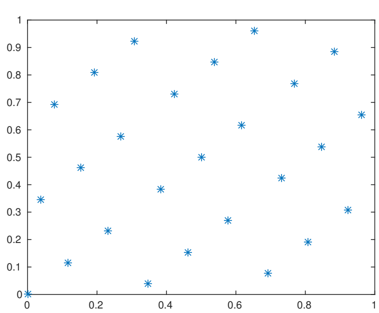

Figure 2: Fixed points of $\mathcal{F}^3$ with standard stacking $(0, 1, 2)$

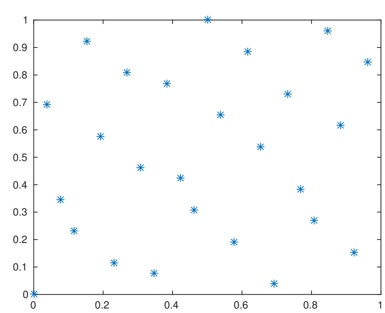

Figure 3: Fixed points of $\mathcal{F}^3$ with stacking $(0, 2, 1)$

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Greg Cossel
USAA
20659 Stone Oak Pkwy #1602
San Antonio TX, 78258
E-mail: `rgcossel@gmail.com`

Werner Horn
Institution California State University, Northridge
18001 Nordhoff St.
Northridge, CA, 91330-8313
E-mail: `werner.horn@csun.edu`

William Sherman
University of California Irvine
Irvine, CA 92697
E-mail: `wsherma1@uci.edu`

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