Grim on Complete Multipartite Graphs

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Abstract - Grim is a deletion game played on the vertices of a graph. In this paper we will examine strategies for when Grim is played on complete multipartite graphs.

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1 Background and Main Theorem

A graph G consists of a set of vertices, V, and a set of edges, E; edges connect pairs of vertices in V. We usually denote a graph $G = (V, E)$. In this article we assume that all graphs have neither multi-edges nor loops. A complete k-partite graph (for $k > 1$) has its vertices partitioned into k sets, and in which edges connect all pairs of vertices that are not on the same partition; no two vertices in the same partition are adjacent to each other. We notate a complete multipartite graph by $K_{n_1,n_2,\cdots,n_t}$ with $n_1 \leq n_2 \leq \cdots \leq n_t$, where a given sub-index represents the number of vertices in a given partition. We will say that a partition with exactly one vertex is a *singleton*. We direct the reader to [3] for more details, and results, about graphs.

A two-player combinatorial game is a game without moves determined by chance, where all possible moves are known to the players (the game has perfect information). Under the assumption that each player will play the "best" possible move, various winning strategies arise within the structure of the game (see [2] for a great study on combinatorial games).

In this article, we are interested in studying a game, called Grim, that is played on graphs. Grim was defined, inspired by the work of Fukuyama in [4] and [5], by Adams et. al in [1], where they studied strategies for this game when played on certain families of graphs. They also wrote a program to compute the Sprague-Grundy values (see [6] and [7]) associated to Grim played on paths; this program computes these values recursively by just implementing the definition of the Sprague-Grundy function.

We explain how to play Grim next.

Definition 1.1 *A legal move of Grim on a graph H consists of deleting a vertex, v, together with all the edges incident with v and any other vertices that may have become isolated after deleting v.*

Given an initial position graph G, Grim is played by two players alternating in taking turns and making one legal move per turn. In the case that G has any isolated vertices, they will be deleted before the first legal move is made. The player making the last legal move is the winner of the graph G.

Let G and H be graphs. If H is obtained from G after a legal Grim move, then we will call H a *follower* of G. If, given a graph H, there is a strategy to win for the next player making a move then we will say that H is an $\mathcal{N}$ position (the $\mathcal{N}$ is because the *next* player will win the game); if the next player to make a move does not have a strategy to win then H is a $\mathcal{P}$ position (the $\mathcal{P}$ is because the *previous* player would now win the game).

In [1], winning strategies for Grim were found for complete tripartite graphs, and complete multipartite graphs having exactly one singleton or no singletons, among other things. The following two results are from that article; the have been re-phrased using the number of vertices of the graph, $|V|$, to make them to read similar to results ahead in this article.

Lemma 1.2 *Let $m, n \in \mathbb{N}$. Then, the following hold.*

(a) *K_n is an $\mathcal{N}$ position if and only if $|V|$ is even.*

(b) *$K_{1,n}$ is an $\mathcal{N}$ position, for all $|V|$.*

(c) *Assume $m, n > 1$. Then, $K_{m,n}$ is an $\mathcal{N}$ position if and only if $|V|$ is odd.*

(d) *$K_{1,1,n}$ is an $\mathcal{N}$ position if and only if $|V|$ is even.*

(e) *Assume $n \geq 2$. Then, $K_{1,2,n}$ is an $\mathcal{N}$ position, for all $|V|$.*

(f) *Assume $m, n \geq 3$. Then, $K_{1,m,n}$ is an $\mathcal{N}$ position if and only if $|V|$ is odd.*

Lemma 1.3 *Let $G = K_{n_1,n_2,\cdots,n_t}$, where $n_i \in \mathbb{N}$, for all $i = 1, \ldots, t$. Then,*

(a) *Assume $t \geq 3$ and $n_i \geq 2$, for all $i = 1, \ldots, t$. Then, G is an $\mathcal{N}$ position if and only if $|V|$ is odd.*

(b) *Assume $n_1 = 1$, $t \geq 4$, and $n_i \geq 3$, for all $i = 2, \ldots, t$. Then, G is an $\mathcal{N}$ position if and only if $|V|$ is odd.*

The results for complete tripartite graphs in Lemma 1.2 motivates us to look further into cases where $\mathcal{N}$ positions determined by $|V|$ even switch to $|V|$ odd. This leads us to look into complete multipartite graphs with at least two singletons.

Notation. In order to help ourselves in the writing of our arguments, we shall write complete multipartite graphs by $K_{n_0^{t_0},n_1^{t_1},n_2^{t_2},\ldots,n_k^{t_k}}$, where $n_0 \leq n_1 \leq \cdots \leq n_k$, and $t_i \in \mathbb{N}$ represents the number of partitions with n_i vertices.

In the following sections we address the problem of playing Grim on a class of graphs that were not considered in Lemma 1.3. These are all the complete multipartite graphs that have at least one partition containing exactly two vertices (note that part *(b)* requires $n_i \geq 3$, for all $i > 1$). All the results in the next two sections are summarized in our Main Theorem, which follows; we will prove our Main Theorem in Section 4.

Theorem 1.4 *Assume that $m, n, s, t \in \mathbb{N}$, $s \geq 0$, $t \geq 2$, and $m, n \geq 2$. Then, the following hold for graphs of the form $K_{1^t,2^s,m,n}$.*

(1) For $m < t+1$, $K_{1^t,2^s,m,n}$ is an $\mathcal{N}$ position if and only if $|V|$ is even.

(2) For $m = t+1$, $K_{1^t,2^s,m,n}$ is an $\mathcal{N}$ position for all $n \geq m$.

(3) For $m > t+1$, $K_{1^t,2^s,m,n}$ is an $\mathcal{N}$ position if and only if $|V|$ is odd.

Using our Main Theorem, along with Lemma 1.3, we are able to quickly determine the outcome of Grim played on complete multipartite graphs of the following types: (1) having no singletons, (2) having exactly one singleton, or (3) consisting of mostly singletons and/or partitions containing two vertices. Note that, for example, the graph $K_{1,1,1,3,3,3}$ is not addressed in either result.

2 Complete Four-Partite Graphs

We begin with an examination of strategies for the "smallest" type of complete multipartite graphs with at least two singletons. Since the strategies for bipartites and tripartites are known, we naturally continue with complete four-partite graphs.

Lemma 2.1 *Let $m, n \in \mathbb{N}$, and $G = K_{1,1,m,n}$. Then,*

(a) $K_{1^3,n}$, where $n \geq 1$, is an $\mathcal{N}$ position if and only if $|V|$ is even.

(b) $K_{1^2,2,n}$, where $n \geq 2$, is an $\mathcal{N}$ position if and only if $|V|$ is even.

(c) $K_{1^2,3,n}$, is an $\mathcal{N}$ position, for all $n \geq 3$.

(d) $K_{1^2,m,n}$, where $m, n \geq 4$, is an $\mathcal{N}$ position if and only if $|V|$ is odd.

Proof. *(a)* First consider the case K_{1^4}. From this starting position, all moves will result in the follower K_{1^3}, a $\mathcal{P}$ position. Thus, K_{1^4} is an $\mathcal{N}$ position. Now we consider the case where $n = 2$. The starting move could give the follower K_{1^4}, an $\mathcal{N}$ position, or $K_{1^2,2}$, which by Lemma 1.2 is also an $\mathcal{N}$ position. Thus, $K_{1^3,2}$ is a $\mathcal{P}$ position.

We proceed by induction on n, where $n \geq 3$; we want to show that $K_{1^3,n+1}$ is an $\mathcal{N}$ position, for $n+1 \geq 4$.

When $n+1$ is odd, Player 1 will play the starting move that yields $K_{1,1,1,n}$. Since n is an even number, by our inductive hypothesis this is a $\mathcal{P}$ position. Hence, in this case, $K_{1^3,n+1}$ is an $\mathcal{N}$ position.

Assume $n+1$ is even. If Player 1 leaves the follower $K_{1^3,n}$ then, as n is odd, K_{1^4} is an $\mathcal{N}$ position by induction. It follows that $K_{1^3,n}$ is a $\mathcal{P}$ position. Now, if Player 1 leaves $K_{1^2,n+1}$ as a follower. Since $n+1$ is even, by Lemma 1.2, this is an $\mathcal{N}$ position. Thus, irrespective of the starting move, $K_{1^3,n+1}$ is a $\mathcal{P}$ position when $n+1$ is even.

(b) First consider the case $K_{1^2,2^2}$. From this starting position, Player 1 will choose the move that results in $K_{1^3,2}$, which by *(a)* is a $\mathcal{P}$ position. Thus, $K_{1^2,2^2}$ is an $\mathcal{N}$ position.

Now we consider the case $K_{1^2,2,3}$. The follower $K_{1,1,2^2}$ is the previous base case, the follower $K_{1^3,3}$ is an $\mathcal{N}$ position by *(a)*, and the follower $K_{1,2,3}$ is an $\mathcal{N}$ position by Lemma 1.2. In either case, $K_{1^2,2,3}$ is a $\mathcal{P}$ position.

Now we proceed by induction on n, where $n \geq 4$; we want to show that $K_{1^2,2,n+1}$ is an $\mathcal{N}$ position, for $n+1 \geq 5$, if and only if $n+1$ is even.

If $n+1$ is even, then Player 1 will choose the starting move that gives the follower $K_{1^3,n+1}$, which, by *(a)*, is a $\mathcal{P}$ position. Thus, $K_{1^2,2,n+1}$ is an $\mathcal{N}$ position when $n+1$ is even.

If $n+1$ is odd, then the follower $K_{1^2,2,n}$ gives the previous case. The follower $K_{1^3,n+1}$ is an $\mathcal{N}$ position by *(a)*, and the follower $K_{1,2,n+1}$ is an $\mathcal{N}$ position by Lemma 1.2. Thus $K_{1^2,2,n+1}$ is a $\mathcal{P}$ position when $n+1$ is odd.

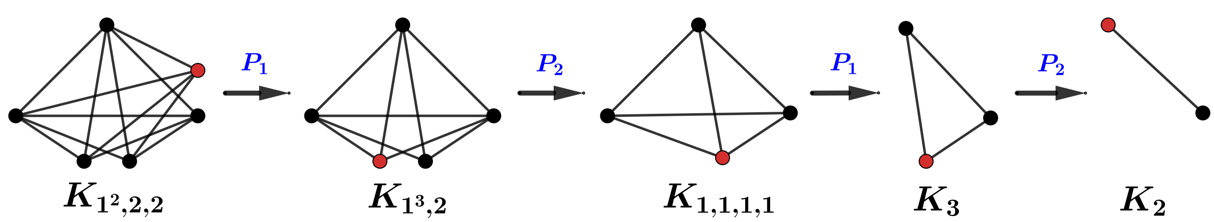
(c) First consider the base case $K_{1^2,3^2}$. Player 1 will choose the follower $K_{1^2,2,3}$, a $\mathcal{P}$ position by a previous case. Thus $K_{1^2,3^2}$ is an $\mathcal{N}$ position.

Next consider $K_{1^2,3,n}$, for n odd. Player 1 will choose the follower $K_{1^2,2,n}$, a $\mathcal{P}$ position by a previous case. When n is even, then Player 1 will choose the follower $K_{1,3,n}$, a $\mathcal{P}$ position by Lemma 1.2. Hence, $K_{1^2,3,n}$ is an $\mathcal{N}$ position for all $n \geq 3$.

(d) First consider the base case $K_{1^2,4^2}$. The follower $K_{1^2,3,4}$ is an $\mathcal{N}$ position by a previous case, while the follower $K_{1,4^2}$ is an $\mathcal{N}$ position by Lemma 1.2. Thus $K_{1^2,4^2}$ is a $\mathcal{P}$ position.

Next consider the case when $m+n$ is odd. Player 1 will choose the follower $K_{1,m,n}$, a $\mathcal{P}$ position by Lemma 1.2. When $m+n$ is even, the follower $K_{1,m,n}$ is an $\mathcal{N}$ position by Lemma 1.2. Any other followers both give $m+n-1$ odd, which is an $\mathcal{N}$ position by the previous case. Hence, $K_{1^2,m,n}$ is a $\mathcal{P}$ position when $m+n$ is even. □

Example 2.2 We consider $K_{1^2,2,2}$, and we play by deleting the vertex in red (in the diagram below). We see that Player 1 will win the graph by taking the last two vertices from K_2.



Hence, $K_{1^2,2,2}$ is an $\mathcal{N}$ position. This is consistent with part *(b)* in Lemma 2.1, as $|V| = 1+1+2+2 = 6$ is even.

3 Complete Multipartite Graphs with Several Partitions

In the previous section we discussed all complete four-partite graphs that had not been addressed in Lemma 1.3. Also, we can see in Lemma 2.1 that there is a pattern of alternation that switched from $\mathcal{N}$ positions being determined by $|V|$ even to $|V|$ odd; this behavior is similar to what we see in Lemma 1.3. In this section, we expand these results to complete multipartite graphs consisting of large numbers of partitions.

Given that most of the results in this section are fairly technical, we provide a short comment on its organization. We start by studying graphs of the form $K_{1^t,2,n}$ in Lemma 3.1, we then generalize this result in Lemma 3.2, where we look at the behavior of $K_{1^t,s,n}$, for $n, s, t \geq 2$. Finally, in Lemma 3.4 we consider a slightly different family of graphs graphs, those of the form at $K_{1,2^t,s,n}$, for $s, t \geq 1$, and $n \geq 2$. After these results have been proved, we will be ready to prove our Main Theorem.

Lemma 3.1 *Let $n, t \in \mathbb{N}$, where $n, t \geq 2$, then $K_{1^t,2,n}$ is an $\mathcal{N}$ position if and only if $|V|$ is even.*

Proof. We will prove this lemma in three stages: we will first consider two particular cases that will serve as 'base cases' for the main proof.
Claim 1. Let $G = K_{1^4,n}$, where $n \in \mathbb{N}$. Then G is an $\mathcal{N}$ position if and only if $|V|$ is even.
Proof of Claim 1. First consider the case where $G = K_{1^5}$. Then, there is only one possible follower K_{1^4}, which is an $\mathcal{N}$ position by Lemma 2.1. Thus, G is a $\mathcal{P}$ position.

If n is even, then Player 1 will choose the follower $K_{1^3,n}$, a $\mathcal{P}$ position by Lemma 2.1. Thus, G is an $\mathcal{N}$ position. If n is odd, then the follower $K_{1^4,n-1}$ is an $\mathcal{N}$ position by a previous case. The other possible follower, $K_{1^3,n}$, is also an $\mathcal{N}$ position by Lemma 2.1. Thus, G is a $\mathcal{P}$ position.

Claim 2. Let $n, t \in \mathbb{N}$, where $n, t \geq 2$, then $K_{1^t,n}$ is an $\mathcal{N}$ position if and only if $|V|$ is even.
Proof of Claim 2. The cases when $t \in \{2, 3\}$ have been already studied in Lemma 1.2 and Lemma 2.1. So, for the rest of the proof we assume that $t \geq 4$ and $n \geq 2$.
($\Leftarrow$) We will proceed by induction on $s = t + n$. If $s = 6$ we get $G = K_{1^4,2}$, which is an $\mathcal{N}$ position by Claim 1.

We assume that $K_{1^a,b}$ is an $\mathcal{N}$ position for some even $s = a + b > 6$ (and $a \geq 4$), and let $G = K_{1^t,n}$ be so that $t + n = s + 2$ (and $t \geq 4$). We will focus our attention on the follower $H = K_{1^{t-1},n}$; its followers are $H_1 = K_{1^{t-2},n}$ and $H_2 = K_{1^{t-1},n-1}$. Since $t + n - 2$ is even, these two graphs are $\mathcal{N}$ positions, by induction and because, when $n = 2$, L is an $\mathcal{N}$ position by Lemma 1.2(a). Hence, H is a $\mathcal{P}$ position, which makes G an $\mathcal{N}$ position.
($\Rightarrow$) Assume that $G = K_{1^t,n}$, where $t + n \geq 4$ is odd. The followers of G are $G_1 = K_{1^{t-1},n}$ and $G_2 = K_{1^t,n-1}$. Since $t + n - 1$ is even, both G_1 and G_2 are $\mathcal{N}$ positions by the other direction of this proof. Hence, G is a $\mathcal{P}$ position.

We finally get to prove the lemma.

($\Leftarrow$) Let $s = t + n$. We will proceed by induction on s. If $s = 4$ we get $G = K_{1^2,2,2}$, which is an $\mathcal{N}$ position by Lemma 2.1. We assume that $K_{1^a,2,b}$ is an $\mathcal{N}$ position for some even $s = a + b \geq 4$. Now consider $G = K_{1^t,2,n}$, with $t + n = s + 2$ and notice that $H = K_{1^{t+1},n}$ is one of its followers. Since $t + n + 1$ is odd, H is a $\mathcal{P}$ position by Claim 2. Hence, G is an $\mathcal{N}$ position.
($\Rightarrow$) Let $t+n > 4$ be odd and consider $G = K_{1^t,2,n}$. The followers of G are $G_1 = K_{1^{t-1},2,n}$, $G_2 = K_{1^{t+1},n}$, and $G_3 = K_{1^t,2,n-1}$. Since $t + n + 1$ and $t + n - 1$ are even, both G_1 and G_3 are $\mathcal{N}$ positions by the other direction of this proof. If $n = 2$, observe that $G_1 = K_{1,2,n}$ is an $\mathcal{N}$ position by Lemma 1.2. For $t = 2$, $G_3 = K_{1^{t+1},2}$ has follower $K_{1^{t+2}}$, with $t + 2$ odd, which is a $\mathcal{P}$ position by Lemma 1.2(a) making G_3 an $\mathcal{N}$ position. G_2 is also an $\mathcal{N}$ position by Claim 2. It follows that G is a $\mathcal{P}$ position. $\square$

The following lemma continues addressing complete multipartite graphs with many singletons.

Lemma 3.2 *The following hold for graphs $K_{1^t,s,n}$, where $n, s, t \in \mathbb{N}$ and $n, s, t \geq 2$.*

(1) For $s < t + 1$, $K_{1^t,s,n}$ is an $\mathcal{N}$ position if and only if $|V|$ is even.

(2) For $s = t + 1$, $K_{1^t,s,n}$ is an $\mathcal{N}$ position for all $n \geq s$.

(3) For $s > t + 1$, $K_{1^t,s,n}$ is an $\mathcal{N}$ position if and only if $|V|$ is odd.

Proof. *(1) $\Leftarrow$)* We will proceed by induction on k, where $2k = t + s + n$. First, we need to discuss a few special cases separately. If n is even then $K_{1^2,2,n}$ is an $\mathcal{N}$ position for n even by Lemma 3.1. If $s = 2$ then $K_{1^t,2,n}$ is an $\mathcal{N}$ position by Lemma 3.1 (as $t + n$ is even). From now on we assume that $s \geq 3$.

Assume that $G = K_{1^a,b,c}$ is an $\mathcal{N}$ position when $6 \leq a+b+c \leq 2k$, and $a, b, c \geq 2$ and $b < a + 1$. Consider $G = K_{1^t,s,n}$, with $t + s + n = 2(k + 1)$ and $s \geq 3$. We will focus our attention on the follower $I = K_{1^t,s-1,n}$, which has followers $I_1 = K_{1^{t-1},s-1,n}$, $I_2 = K_{1^t,s-2,n}$, and $I_3 = K_{1^t,s-1,n-1}$. Note that all of them have $2k$ vertices. Since $s - 1 < t + 1$ and $s - 1 \geq 2$, we get that I_3 is an $\mathcal{N}$ position by our hypothesis. Similarly, if $s > 3$ then I_2 is also an $\mathcal{N}$ position, as $s - 2 < t + 1$ and $s - 2 \geq 2$. For $s = 3$, $I_2 = K_{1^{t+1},n}$, which is an $\mathcal{N}$ position by Lemma 3.1 (as $t + 1 + n$ is even). Finally for I_1, since $s - 1 < (t - 1) + 1$ and $s - 1 \geq 2$, it is also an $\mathcal{N}$ position. It follows that I is a $\mathcal{P}$ position, and thus G is an $\mathcal{N}$ position.

(3) $\Leftarrow$) We will proceed by induction on k, where $2k+1 = t+s+n$. Firstly, we note that, for $t = 2$ and n odd, $K_{1^2,4,n}$ is an $\mathcal{N}$ position by Lemma 2.1. From now on we assume that $t \geq 3$.

Assume that $G = K_{1^a,b,c}$ is an $\mathcal{N}$ position when $7 \leq a + b + c \leq 2k - 1$, and $a, b, c \geq 2$ with $b > a + 1$. Consider $G = K_{1^t,s,n}$, with $t + s + n = 2k + 1$ and $t \geq 3$. We will focus our attention on the follower $H = K_{1^{t-1},s,n}$, which has followers $H_1 = K_{1^{t-2},s,n}$, $H_2 = K_{1^{t-1},s-1,n}$, and $H_3 = K_{1^{t-1},s,n-1}$. Note that all of these graphs have $2k-1$ vertices. If $t > 3$, we get that H_1 is an $\mathcal{N}$ position by our hypothesis since $s > (t - 2) + 1$ and

$t-2 \geq 2$. For $t=3$, $H_1 = K_{1,s,n}$ with $s \geq 2$ and $s+n$ even, which is an $\mathcal{N}$ position by Lemma 1.2. Similarly, since $s \geq 3$ then H_2 is also an $\mathcal{N}$ position, as $s-1 > (t-1)+1$ and $s-1 \geq 2$. Finally, since $n \geq 3$ because partitions are listed in increasing order, H_3 is also an $\mathcal{N}$ position as $s > t$ and $n-1 \geq 2$. It follows that H is a $\mathcal{P}$ position, and thus G is an $\mathcal{N}$ position.

(2) We will proceed by induction on $v = 2t+n+1$. But first we remark on the fact that $K_{1^2,3,n}$ is an $\mathcal{N}$ position, for all n, by Lemma 2.1. So, from now on, we only need to prove our claim for $t \geq 3$. As usual, the sizes of the partitions are listed in increasing order, and so $t+1 \leq n$.

Assume that $K_{1^a,a+1,c}$ is an $\mathcal{N}$ position when $8 \leq 2a+c+1 \leq v$, where $a \geq 2$. Consider $G = K_{1^t,t+1,n}$ such that $2t+n+1 = v+2$. The followers of G are $H = K_{1^{t-1},t+1,n}$, $I = K_{1^t,t,n}$, and $J = K_{1^t,t+1,n-1}$.

Case 1: If v is odd, we assume Player 1 leaves H as a follower. The followers of H are $H_1 = K_{1^{t-2},t+1,n}$, $H_2 = K_{1^{t-1},t,n}$, and $H_3 = K_{1^{t-1},t+1,n-1}$; all of them having v vertices.

- For H_1: The graphs $K_{1,4,n}$ ($t=3$ and n even) and $K_{1^2,5,n}$ ($t=4$ and n even) are $\mathcal{N}$ positions by Lemma 1.2 and Lemma 2.1, respectively. For $t \geq 5$, since $t+1 > (t-2)+1$ and v is odd, H_1 is an $\mathcal{N}$ position by *(3 $\Leftarrow$)*.

- H_2 is an $\mathcal{N}$ position by our inductive hypothesis, together with $t-1 \geq 2$.

- For H_3: Note that if $t+1 > n-1$ the graph is $H_3 = K_{1^{t-1},n-1,t+1}$. However, since $t+1 \leq n$, we must have that $(t-1)+1 = n-1$. In this case, H_3 is an $\mathcal{N}$ position by our inductive hypothesis. For $t+1 \leq n-1$, since $t+1 > (t-1)+1$, $t-1 \geq 2$, $n-1 \geq 2$, and v is odd, H_3 is an $\mathcal{N}$ position by *(3 $\Leftarrow$)*.

Thus, H is a $\mathcal{P}$ position, making G an $\mathcal{N}$ position.

Case 2: If v is even, we assume Player 1 leaves I as a follower. The followers of I are $I_1 = K_{1^{t-1},t,n}$, $I_2 = K_{1^t,t-1,n}$, and $I_3 = K_{1^t,t,n-1}$; all of them having v vertices.

- I_1 is an $\mathcal{N}$ position by our inductive hypothesis, together with $t-1 \geq 2$.

- For I_2: Since $2 \leq t-1 < t+1$ and v is even, I_2 is an $\mathcal{N}$ position by *(1 $\Leftarrow$)*.

- For I_3: Just like in the proof of *(1 $\Leftarrow$)*, we need to look at the case when the last two subindexes 'switch'. This situation reduces to us having to look at when $n = t+1$; we get that $K_{1^t,t,t}$ is an $\mathcal{N}$ position by *(1 $\Leftarrow$)*. When $t \leq n-1$, we get that I_3 is an $\mathcal{N}$ position by *(1 $\Leftarrow$)*, as $t < t+1$ and v is even.

It follows that I is a $\mathcal{P}$ position, making G an $\mathcal{N}$ position.

(1 $\Rightarrow$) Let $s < t+1$ and $|V|$ be odd. If $s = t$ then G_1 is an $\mathcal{N}$ position, as $s = (t-1)+1$, by *(2)*; if $s < t$ then G_1 is still an $\mathcal{N}$ position by the proof of *(1 $\Leftarrow$)*, as $s < (t-1)+1$ (the subcase $t = 2$ uses Lemma 1.2 to get to the same conclusion since it would make $s \leq 2$). Now for G_2. If $s = 2$, then $t+n+1$ is even and we use Lemma 3.1 to get that G_2 is an

$\mathcal{N}$ position. If $s > 2$, since $s - 1 < t + 1$, G_2 is an $\mathcal{N}$ position by *(1 $\Leftarrow$)*. Finally, for G_3. If $n = 2$, we get $G_3 = K_{1^{t+1},s}$ with $t + s + 1$ even and use Lemma 3.1 to show that G_3 is an $\mathcal{N}$ position. When $n > 2$, we use that $s < t + 1$ to get that G_3 is an $\mathcal{N}$ position by *(1 $\Leftarrow$)*. If $s = n$, we can rewrite $G_3 = K_{1^t,n-1,s}$ which is still an $\mathcal{N}$ position much like G_2. It follows that G is a $\mathcal{P}$ position.

(3) $\Rightarrow$) Let $s > t + 1$ and $|V|$ to be even. For G_1, if $t = 2$ then G_1 is an $\mathcal{N}$ position by Lemma 1.2. When $t > 2$, we get that $s > (t - 1) + 1$, and so G_1 is an $\mathcal{N}$ position by the other direction of *(3)*. Now, when $s = t + 2$ we get that $G_2 = K_{1^t,t+1,n}$ is an $\mathcal{N}$ position by *(2)*, whereas if $s > t + 2$ then G_2 is still an $\mathcal{N}$ position by the other direction of *(3)* (since $s > t + 1 \geq 3$, we do not need to address $s = 2$). Likewise, since partitions are listed in increasing order, $n \geq s \geq 3$. Finally, if $s = n$, we treat $G_3 = K_{1^t,n-1,s}$ as G_2, and if $s < n$ we use that $s > t + 1$ to get G_3 to be an $\mathcal{N}$ position by the other direction of *(3)*. Hence, G is a $\mathcal{P}$ position. $\square$

Notice that the winner of most of the graphs considered in Lemma 3.2 depends on $|V|$ being even or odd. We next continue exploring this alternating pattern, now looking at graphs that contain several partitions containing exactly two vertices.

Lemma 3.3 *For $n \in \mathbb{N}$, where $n \geq 2$, $K_{1^2,2^2,n}$ is an $\mathcal{N}$ position if and only if $|V|$ is even.*

Proof. ($\Leftarrow$) Consider $G = K_{1^2,2^2,n}$, with any even $n \geq 2$ and notice that $H = K_{1^3,2,n}$ is a follower. We can use Lemma 3.1 with $t + n$ odd to get that H is a $\mathcal{P}$ position, and thus G is an $\mathcal{N}$ position.
($\Rightarrow$) Now consider $G = K_{1^2,2^2,n}$, where $n > 2$ is odd. It is easy to see that G will have three followers: $G_1 = K_{1,2^2,n}$, $G_2 = K_{1^3,2,n}$, and $G_3 = K_{1^2,2^2,n-1}$, of which the last two are $\mathcal{N}$ positions by Lemma 3.1 and the other direction of this proof, respectively. For G_1, the follower $K_{1^2,2,n}$ is a $\mathcal{P}$ position by Lemma 2.1, thus making G_1 an $\mathcal{N}$ position. It follows that G is a $\mathcal{P}$ position. $\square$

The previous lemma will be useful to prove the following, very important, lemma.

Lemma 3.4 *Assume that $n, s, t \in \mathbb{N}$, $s, t \geq 1$, and $n \geq 2$. Then, the following hold for graphs $K_{1,2^t,s,n}$:*

(1) For $s < 2$, $K_{1,2^t,s,n}$ is an $\mathcal{N}$ position if and only if $|V|$ is even.

(2) For $s = 2$, $K_{1,2^t,s,n}$ is an $\mathcal{N}$ position for all $n \geq s$.

(3) For $s > 2$, $K_{1,2^t,s,n}$ is an $\mathcal{N}$ position if and only if $|V|$ is odd.

Proof. We start by looking at when $t = 1$. These follow because $G = K_{1^2,2,n}$, with $|V|$ even, is an $\mathcal{N}$ position by Lemma 2.1. $G = K_{1,2,s,n}$, with $|V|$ odd, has the follower $K_{2,s,n}$ which is a $\mathcal{P}$ position by Lemma 1.3, making it an $\mathcal{N}$ position as well. Finally, $K_{1,2,n}$ is an $\mathcal{N}$ position by Lemma 1.2. From now on we will consider $t > 1$.

(1) ⇒) We start by noticing that $K_{1^2,2^2,n}$, for odd $n > 2$, is a $\mathcal{P}$ position by Lemma 3.3. We now assume that for some $m, n, t \geq 2$, and $|V(G)| < m$ (m odd) that $G = K_{1^2,2^t,n}$ is a $\mathcal{P}$ position when n is odd. Consider $H = K_{1^2,2^a,b}$ with b odd and $|V(H)| = m$. This graph has followers: $H_1 = K_{1,2^a,b}$, $H_2 = K_{1^3,2^{a-1},b}$, and $H_3 = K_{1^2,2^a,b-1}$. Doing induction on $a+b$, with b odd, we get that H_1 has the follower $K_{1^2,2^{a-1},b}$ and H_2 has the follower $K_{1^2,2^{a-2},b}$, both of which are $\mathcal{P}$ positions by our hypothesis. Thus, H_1 and H_2 are $\mathcal{N}$ positions. For H_3, we use our hypothesis with $b-2$ being odd to get that the follower $K_{1^2,2^a,b-2}$ is a $\mathcal{P}$ position, making H_3 an $\mathcal{N}$ position as well. It follows that H is a $\mathcal{P}$ position.

(1) ⇐) Now consider $G = K_{1^2,2^t,n}$ with $n, t \geq 2$ and n even. Observe that G has the follower $K_{1^2,2^t,n-1}$, where $n-1$ is odd, which is a $\mathcal{P}$ position by the other direction of this proof. Hence, G is an $\mathcal{N}$ position.

(2) Assume $G = K_{1,2^t,n}$, with $n, t \geq 2$ and $|V(G)| < m$ is an $\mathcal{N}$ position, for all $n \geq 2$. Let $H = K_{1,2^a,b}$ with $|V(H)| = m$. If b is even, we use Lemma 1.3 to get that H has a follower $K_{2^a,b}$, with $|V|$ even, that is a $\mathcal{P}$ position. If b is odd, H has the follower $K_{1^2,2^{a-1},b}$, which is a $\mathcal{P}$ position by Lemma 2.1, for $a = 2$, and by *(1)*, for $a > 2$. Hence, H is also an $\mathcal{N}$ position.

(3) ⇐) Observe that for $t \geq 2$ and $n, s > 2$, any graph $G = K_{1,2^t,s,n}$ with $|V|$ odd will have the follower $K_{2^t,s,n}$ which is a $\mathcal{P}$ position by Lemma 1.3. So G is an $\mathcal{N}$ position.

(3) ⇒) Consider the base case $G = K_{1,2^2,3,n}$ with $|V|$ even, which has followers $K_{2^2,3,n}$ and $K_{1,2^3,n}$; these are both $\mathcal{N}$ positions by Lemma 1.3 and *(2)*, respectively. The follower $K_{1^2,2,3,n}$ has its own follower $K_{1^3,3,n}$ that is a $\mathcal{P}$ position by Lemma 3.2, making it another $\mathcal{N}$ position. The last follower $K_{1,2^2,3,n-1}$, also has its own follower $K_{2^2,3,n-1}$ that is a $\mathcal{P}$ position by Lemma 1.3, making it an $\mathcal{N}$ position as well. It follows that G is a $\mathcal{P}$ position.

Now, for $t \geq 2$ and $n, s > 2$, take as induction hypothesis that $G = K_{1,2^t,s,n}$, with $|V(G)|$ even, is a $\mathcal{P}$ position. Consider $H = K_{1,2^{t+1},s,n}$ with $|V(H)| = m$ even, will have the followers:

- $K_{2^{t+1},s,n}$. Since $|V| = m-1$ is odd, this is an $\mathcal{N}$ position by Lemma 1.3.
- $K_{1^2,2^t,s,n}$. This graph has $K_{1,2^t,s,n}$ as a follower, which by our hypothesis is a $\mathcal{P}$ position, making $K_{1^2,2^t,s,n}$ an $\mathcal{N}$ position.
- $K_{1,2^{t+1},s-1,n}$ and $K_{1,2^{t+1},s,n-1}$. Since m is odd, both of these graphs are $\mathcal{N}$ positions by the other direction of this proof, even when $s = n$. However, if $s = 3$ or $n = 3$ we have that these graphs are $\mathcal{N}$ positions by *(2)* instead.

Since all followers of H are $\mathcal{N}$ positions, it follows that H is a $\mathcal{P}$ position. □

Since the results of Lemma 3.4 closely resemble those in Lemma 3.2, we continue our work on trying to generalize, and unify, these results.

Theorem 3.5 *Assume that $n, s, t \in \mathbb{N}$, $t \geq 1$, and $s, n \geq 2$. Then, the following hold for $K_{1^2,2^t,s,n}$.*

(1) For $s < 3$, $K_{1^2,2^t,s,n}$ is an $\mathcal{N}$ position if and only if $|V|$ is even.

(2) For $s = 3$, $K_{1^2,2^t,s,n}$ is an $\mathcal{N}$ position for all $n \geq s$.

(3) For $s > 3$, $K_{1^2,2^t,s,n}$ is an $\mathcal{N}$ position if and only if $|V|$ is odd.

Proof. *(1) $\Leftarrow$)* Observe that $G = K_{1^2,2^t,s,n}$, with $s < 3$ and $|V|$ even, is either the graph $G_1 = K_{1^3,2^t,n}$ or $G_2 = K_{1^2,2^{t+1},n}$. Since G_1 has the follower $K_{1^2,2^t,n}$, which is a $\mathcal{P}$ position by Lemma 3.4, G_1 is an $\mathcal{N}$ position. Similarly, G_2 is an $\mathcal{N}$ position by Lemma 3.4. Hence, G is an $\mathcal{N}$ position.

(1) $\Rightarrow$) We start with a base case $G = K_{1^3,2,n}$, for any even $n \geq 2$. The first follower of G, $G_1 = K_{1^2,2,n}$ is an $\mathcal{N}$ position by Lemma 2.1. The second follower, $G_2 = K_{1^4,n}$, is an $\mathcal{N}$ position by Lemma 3.1. The third follower, $G_3 = K_{1^3,2,n-1}$, itself has a follower, $K_{1^2,2,n-1}$, which is a $\mathcal{P}$ position by Lemma 2.1 and is thus an $\mathcal{N}$ position.

Now assume that $G = K_{1^2,2^t,s,n}$, with $s < 3$ and $|V| < m$, is a $\mathcal{P}$ position when $|V|$ is odd. Then, for $H = K_{1^2,2^a,b,c}$, with $|V(H)| = m$ odd, we have the followers $H_1 = K_{1,2^a,b,c}$, $H_2 = K_{1^3,2^a-1,b,c}$, $H_3 = K_{1^2,2^a,b-1,c}$, and $H_4 = K_{1^2,2^a,b,c-1}$.

- H_1 is an $\mathcal{N}$ position by Lemma 3.4, for both $b = 1$ and $b = 2$.
- H_2 has the follower $K_{1^2,2^{a-1},b,c}$ which is a $\mathcal{P}$ position by our induction hypothesis, which makes H_2 an $\mathcal{N}$ position.
- H_3 and H_4 are $\mathcal{N}$ positions by the other direction of this proof, as they have an even number of vertices.

Since all followers of H are $\mathcal{N}$ positions, it follows that H is a $\mathcal{P}$ position.

(2) When n is even, G has the follower $K_{1,2^t,3,n}$, which is a $\mathcal{P}$ position by Lemma 3.4. When n is odd, G has the follower $K_{1^2,2^{t+1},n}$, which is a $\mathcal{P}$ position by *(1)*. In either case, G is an $\mathcal{N}$ position.

(3) $\Leftarrow$) Observe that $G = K_{1^2,2^t,s,n}$ with $3 < s \leq n$ and $|V|$ odd has $K_{1,2^t,s,n}$ as a follower, which is a $\mathcal{P}$ position by Lemma 3.4, so G is an $\mathcal{N}$ position.
(3) $\Rightarrow$) We start with a base case $G = K_{1^2,2,4,n}$, for any even $n \geq 4$, which has the followers $G_1 = K_{1,2,4,n}$, $G_2 = K_{1^3,4,n}$, $G_3 = K_{1^2,2,3,n}$, and $G_4 = K_{1^2,2,4,n-1}$. G_1, G_3, and G_4 are $\mathcal{N}$ positions by Lemma 3.4, *(2)*, and the other direction of this proof, respectively. G_2 has a follower $K_{1^2,4,n}$ that is a $\mathcal{P}$ position by Lemma 2.1. It then follows that G is a $\mathcal{P}$ position since all its followers are $\mathcal{N}$ positions. Now assume that $G = K_{1^2,2^t,s,n}$, with $s > 3$ and $|V| < m$, is a $\mathcal{P}$ position when $|V|$ is even. Then, for $H = K_{1^2,2^a,b,c}$ with $|V(H)| = m$ even, we have the followers $H_1 = K_{1,2^a,b,c}$, $H_2 = K_{1^3,2^a-1,b,c}$, $H_3 = K_{1^2,2^a,b-1,c}$, and $H_4 = K_{1^2,2^a,b,c-1}$.

- H_1 is an $\mathcal{N}$ position by Lemma 3.4.
- H_2 has the follower $K_{1^2,2^{a-1},b,c}$ which is a $\mathcal{P}$ position by our induction hypothesis, so H_2 is an $\mathcal{N}$ position.
- H_3 and H_4 are $\mathcal{N}$ positions by the other direction of this proof, as they have an even number of vertices.

Since all the followers of H are $\mathcal{N}$ positions, H is a $\mathcal{P}$ position. □

4 Proof of Main Theorem

In this section, we prove Theorem 1.4 by using results in previous sections. We first generalize Lemmas 3.4 and 3.5.

Lemma 4.1 *Assume that $m, n, s, t \in \mathbb{N}$, $s, t \geq 1$, and $m, n \geq 2$. Then, the following hold for graphs $K_{1^t,2^s,m,n}$.*

(1) For $m < t+1$, $K_{1^t,2^s,m,n}$ is an $\mathcal{N}$ position if and only if $|V|$ is even.

(2) For $m = t+1$, $K_{1^t,2^s,m,n}$ is an $\mathcal{N}$ position for all $n \geq m$.

(3) For $m > t+1$, $K_{1^t,2^s,m,n}$ is an $\mathcal{N}$ position if and only if $|V|$ is odd.

Proof. We will prove most of these statements by induction on $|V|$.

(1) $\Leftarrow$) Observe that the base case $K_{1^2,2^4}$, where $m, n = 2$ and $|V| = 10$, is an $\mathcal{N}$ position by Lemma 3.4. Now assume that $K_{1^t,2^s,m,n}$, with $m < t+1$, is an $\mathcal{N}$ position, for $10 \leq |V| \leq v-2$, when $|V|$ is even. Then $G = K_{1^a,2^b,c,d}$, with $|V| = v$ and $c < a+1$, has the follower $H = K_{1^a,2^b,c-1,d}$. We look at the followers of H:

- $H_1 = K_{1^{a-1},2^b,c-1,d}$ is an $\mathcal{N}$ position by our induction hypothesis, and by Theorem 3.5 for $a = 2$.
- $H_2 = K_{1^a,2^{b-1},c-1,d}$, $H_3 = K_{1^a,2^b,c-2,d}$, and $H_4 = K_{1^a,2^b,c-1,d-1}$ are $\mathcal{N}$ positions by our induction hypothesis.

Hence, H is a $\mathcal{P}$ position and G is an $\mathcal{N}$ position.

(3) $\Leftarrow$) Observe that the base case $K_{1^2,2^2,4,5}$, where $|V| = 15$, is an $\mathcal{N}$ position by Lemma 3.4. Now assume that $K_{1^t,2^s,m,n}$, with $m > t+1$, is an $\mathcal{N}$ position, for $15 \leq |V| \leq v-2$ when $|V|$ is odd. Then $G = K_{1^a,2^b,c,d}$, with $|V| = v$ and $c > a+1$, has the follower $J = K_{1^{a-1},2^b,c,d}$. We look at the followers of J:

- $J_1 = K_{1^{a-2},2^b,c,d}$ is an $\mathcal{N}$ position by our induction hypothesis, and by Lemma 1.3 for $a = 2$.

- $J_2 = K_{1^{a-1},2^{b-1},c,d}$, $J_3 = K_{1^{a-1},2^b,c-1,d}$, and $J_4 = K_{1^{a-1},2^b,c,d-1}$ are $\mathcal{N}$ positions by our induction hypothesis.

Hence, J is a $\mathcal{P}$ position and G is an $\mathcal{N}$ position.

(1) $\Rightarrow$) Take $G = K_{1^t,2^s,m,n}$ with $|V|$ odd, $s, t \geq 1$, $m, n \geq 2$, and $m < t+1$. Note that all the followers of G have an even number of vertices, and thus if $m < t+1$ we get that the follower $G_1 = K_{1^t,2^s,m-1,n}$ is an $\mathcal{N}$ position by *(1) $\Leftarrow$)*. Similarly, if $m < t$, the follower $G_2 = K_{1^{t-1},2^s,m,n}$ is also an $\mathcal{N}$ position by *(1) $\Leftarrow$)*. Note that the follower $G_2 = K_{1^{t-1},2^s,t,n}$ (when $m = t > 1$) is also an $\mathcal{N}$ position by *(1) $\Leftarrow$)*; the case $t = 1$ yields both $m \geq 2$, and $m < 2$, which is a contradiction. Now, the follower $G_3 = K_{1^t,2^{s-1},m,n}$ is an $\mathcal{N}$ position by *(1) $\Leftarrow$)* (for $s > 1$); the case when $s = 1$ yields $G_3 = K_{1^t,m,n}$, which is an $\mathcal{N}$ position by Lemma 3.2. Finally, the follower $G_4 = K_{1^t,2^s,m,n-1}$ is an $\mathcal{N}$ position by *(1) $\Leftarrow$)* (for $n > 2$); the case when $n = 2$ yields $G_4 = K_{1^{t+1},2^s,m}$, which is an $\mathcal{N}$ position by Lemma 3.2.

(3) $\Rightarrow$) follows in almost the same way *(1) $\Rightarrow$)* did.

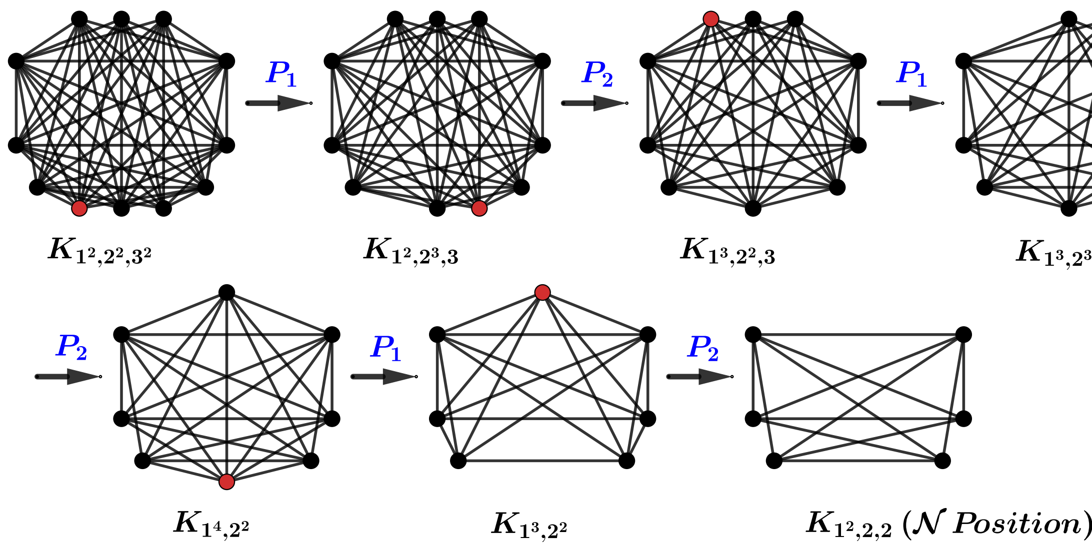
(2) Note that $K_{1^2,2,3,n}$ is an $\mathcal{N}$ position by Lemma 3.4. Now assume $K_{1^t,2^s,t+1,n}$ (where the smallest $|V|$ can be is 10) is an $\mathcal{N}$ position, for all n and for all $10 \leq |V| < v$, for some $v \in \mathbb{N}$. Now consider $G = K_{1^a,2^b,a+1,d}$ be so that $|V(G)| = v$.

Suppose that v is odd. We look at the follower $H = K_{1^{a-1},2^b,a+1,d}$ and its own followers: $H_1 = K_{1^{a-2},2^b,a+1,d}$, $H_2 = K_{1^{a-1},2^{b-1},a+1,d}$, $H_3 = K_{1^{a-1},2^b,a,d}$, and $H_4 = K_{1^{a-1},2^b,a+1,d-1}$. Since H_1, H_2, H_4 have an odd number of vertices and $a+1 > a$, they are $\mathcal{N}$ positions by *(3) $\Leftarrow$)*. By our induction hypothesis, H_3 is also an $\mathcal{N}$ position, which makes H a $\mathcal{P}$ position. Hence, G is an $\mathcal{N}$ position.

Now suppose that v is even. We focus our attention on the follower $J = K_{1^a,2^b,a,d}$ and on its followers: $J_1 = K_{1^{a-1},2^b,a,d}$, $J_2 = K_{1^a,2^{b-1},a,d}$, $J_3 = K_{1^a,2^b,a-1,d}$, and $J_4 = K_{1^a,2^b,a,d-1}$. Since J_2, J_3, J_4 have an even number of vertices and $a < a+1$, they are $\mathcal{N}$ positions by *(1) $\Leftarrow$)*. By our induction hypothesis, J_1 is also an $\mathcal{N}$ position, making J a $\mathcal{P}$ position. Thus, G is an $\mathcal{N}$ position. □

Since our Main Theorem simply merges the results in Lemma 4.1 with Lemma 3.2, we have finished its proof. Next we illustrate, with an example, how to play Grim on a graph that is considered in our Main Theorem.

Example 4.2 We consider $K_{1^2,2^2,3^2}$, and we play by deleting the vertex in red (in the diagram below). We see that after a few moves we get $K_{1^2,2,2}$, which is an $\mathcal{N}$ position by Example 2.2. Since at that point the next player to make a move is Player 1, we get that $K_{1^2,2^2,3^2}$ is won by Player 1, and thus it is also an $\mathcal{N}$ position. This result is consistent with part *(2)* in Theorem 1.4, as $m = 3 = 2 + 1 = t + 1$.



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