On Classical Multiplier Sequences

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Abstract - We provide the detailed proofs of two recent claims of Csordas and Forgács asserting that two particular Bessel-type functions generate classical multiplier sequences whose generic terms are Cauchy-products of Laguerre polynomials and hypergeometric functions, respectively. The way these sequences are generated involves functions from the Laguerre-Pólya class. We address the question whether these functions are unique in any way as generators of a given classical multiplier sequence.

Keywords : Multiplier sequences; Laguerre-Pólya class

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1 Introduction

Any sequence of real numbers $\{\gamma_n\}_{n=0}^{\infty}$ induces a linear operator $\Gamma : \mathbb{R}[x] \to \mathbb{R}[x]$ by setting $\Gamma[x^n] = \gamma_n x^n$ for all $n \in \mathbb{N} \cup \{0\}$, and extending the action to all of $\mathbb{R}[x]$ linearly. The study of the various properties of such operators – in particular whether they map the set of real rooted polynomials into itself – is still an active area of research, despite its over one hundred year-old origins dating back to Laguerre and Hermite. Although the Pólya-Schur program (originated by G. Pólya and J. Schur in their seminal 1914 paper [8]) has culminated in the 2009 paper [1] of J. Borcea and P. Brändén, which characterizes linear operators preserving classes of polynomials with zero loci in circular domains, certain interesting questions related to such operators (and the sequences they represent) remain unsettled. The following are two open problems of interest:

Problem A: Identify explicit defining characteristics of functions $f : \mathbb{R} \to \mathbb{R}$ for which the operator associated to $\{\gamma_n\}_{n=0}^{\infty} = \{f(n)\}_{n=0}^{\infty}$ maps the set of real rooted polynomials into itself.

We hasten to remark that a function f has this property if and only if at least one of $\varphi(x), \varphi(-x), -\varphi(x)$ or $-\varphi(-x)$ belongs to the class $\mathcal{L} - \mathcal{P}^+$ (cf. Definition 2.1), where $\varphi(x) = \sum_{n=0}^{\infty} f(n)/n! x^n$. This characterization however is not easy to use. What Problem A is asking about are properties of such functions that are easier to check, such as having

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only real zeros, or having a certain order and type, for example. Laguerre's Theorem is one result in this direction, but in general, it is not even clear what types of functions one should be considering. The intimate connection of reality preserving sequences to a certain class of real entire functions (namely the Laguerre-Pólya class, cf. Definition 2.1) seems to suggest that one perhaps should consider only real entire functions. Alas, even with such an added restriction there is no known answer to this problem. In addition, there are sequences involving functions that are not entire, and yet either conjectured or proven to be reality preserving (see [3]), such as $\{\sqrt{n}/n!\}_{n=0}^{\infty}$, $\{\ln(n+2)/n!\}_{n=0}^{\infty}$, and $\{[\ln(n+2)+\int_{n+2}^{\infty}\{t\}/t^2 dt]/n!\}_{n=0}^{\infty}$.

Problem B: Give a systematic way to generate reality preserving sequences, i.e. sequences $\{\gamma_n\}_{n=0}^{\infty}$ whose associated operator Γ maps the class of real rooted polynomials into itself (we call these sequences *multiplier sequences*, cf. Definition 2.3).

The set of all multiplier sequences is a monoid under the operation

$$\{\gamma_n\}_{n=0}^{\infty} * \{\beta_n\}_{n=0}^{\infty} = \{\gamma_n\beta_n\}_{n=0}^{\infty}.$$

In particular, the Hadamard product of two multiplier sequences is again a multiplier sequence. Thus, we get a somewhat unsatisfying answer to Problem B: Every multiplier sequence can be decomposed into a product of two multiplier sequences (neither of which is the identity of the monoid), since $\{\gamma_n\}_{n=0}^{\infty} = \{r^n\gamma_n\}_{n=0}^{\infty} * \{(1/r)^n\}_{n=0}^{\infty}$, where $r \in \mathbb{R}\setminus\{0\}$. This fact perhaps also explains why researchers have looked in other directions when trying to understand how to obtain families of multiplier sequences from known multiplier sequences. In a recent paper [3], the authors investigate this question, and suggest an approach that addresses the problem by looking at elements of the Laguerre-Pólya class, rather than at multiplier sequences directly. By doing so, the authors create large families of classical multiplier sequences, parametrized by one or two parameters. In particular, they show that if φ, Φ are elements of the Laguerre-Pólya class with non-negative Taylor coefficients, and $s, t \in [0,1]$, then the sequence $\{C_n^{\varphi,\Phi}(t,s)\}_{n=0}^{\infty}$ generated by the relation

$$e^{(2-s-t)x}\varphi(tx)\Phi(sx) = \sum_{n=0}^{\infty} \frac{C_n^{\varphi,\Phi}(t,s)}{n!}x^k$$

is a classical multiplier sequence.

The goal of this paper is to verify by direct computation two claims made in [3] regarding the explicit form of the sequence $C_n^{\varphi,\Phi}(t,s)$ for particular choices of φ, Φ (see Theorems 3.5 and 3.8). After the verification of these claims, we set out to investigate the extent to which the functions φ and Φ giving rise to a sequence $\{C_n^{\varphi,\Phi}(t,s)\}_{n=0}^{\infty}$ are unique.

2 Background

In order to be able to formulate the claims referred to in the introduction, we need to introduce some definitions and main results in the theory of multiplier sequences. We begin with the definition of a class of functions which play a central role in the theory.

Definition 2.1 *A real entire function* $\psi(x) = \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} x^k$ *is said to belong to the Laguerre-Pólya class, denoted by* $\psi \in \mathscr{L} - \mathscr{P}$*, if it can be represented as*

$$\psi(x) = cx^m e^{-ax^2+bx} \prod_{k=1}^{\omega} \left(1 + \frac{x}{x_k}\right) e^{-x/x_k}$$

where $b, c \in \mathbb{R}, x_k \in \mathbb{R}, m \in \mathbb{Z}^+ \cup \{0\}, a \geq 0, 0 \leq \omega \leq \infty$ *and* $\sum_{k=1}^{\omega} \frac{1}{x_k^2} < +\infty$*. If, in addition,* $\gamma_k \geq 0$ *for all* $k \geq 0$*, we say that* $\psi \in \mathscr{L} - \mathscr{P}^+$*.*

We remark that $\mathscr{L} - \mathscr{P}$ is precisely the class of real entire functions $\sum_{k=0}^{\infty} \frac{a_k}{k!} x^k$, which are locally uniform limits of real polynomials with only real zeros (see for example [6, Satz 9.2])

Definition 2.2 *A real entire function* $\psi(x) = \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} x^k$ *is said to be type I in the Laguerre-Pólya class, denoted by* $\psi \in \mathscr{L} - \mathscr{P}I$*, if* $\psi(x)$*, or* $\psi(-x)$*, can be represented as*

$$\psi(x) = cx^m e^{\sigma x} \prod_{k=1}^{\omega} \left(1 + \frac{x}{x_k}\right),$$

where $c \in \mathbb{R}$*,* m *is a non-negative integer,* $\sigma \geq 0$*,* $x_k > 0$*,* $0 \leq \omega \leq \infty$ *and* $\sum_{k=1}^{\omega} \frac{1}{x_k} < +\infty$*.*

Definition 2.3 *A sequence of real numbers,* $\{\gamma_k\}_{k=0}^{\infty}$*, is called a classical multiplier sequence (of the first kind) if the associated linear operator* $\Gamma : \mathbb{R}[x] \to \mathbb{R}[x]$ *defined by* $\Gamma[x^k] = \gamma_k x^k$ *for* $k = 0, 1, 2, \ldots$*, satisfies*

$$Z_{\mathbb{C}}(\Gamma[p(x)]) = 0 \text{ whenever } Z_{\mathbb{C}}(p(x)) = 0, \qquad \forall\ p(x) \in \mathbb{R}[x],$$

where $Z_{\mathbb{C}}[p(x)]$ *denotes the number of non-real zeros of the polynomial* $p(x)$ *counting multiplicity, and* $Z_{\mathbb{C}}[0] = 0$*.*

The following theorem is due to G. Pólya and J. Schur, and characterizes classical multiplier sequences of the first kind.

Theorem 2.4 *(see [8]) Let* $\{\gamma_k\}_{k=0}^{\infty}$ *be a sequence of non-negative terms. The following are equivalent:*

(i) $\{\gamma_k\}_{k=0}^{\infty}$ *is a classical multiplier sequence;*

(ii) $\Gamma[(1+x)^n] \in \mathscr{L}-\mathscr{P}^+ \ \forall n$;

(iii) $\Gamma[e^x] = \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} x^k \in \mathscr{L}-\mathscr{P}^+$.

Remark 2.5 The reader will note that Theorem 2.4 refers to sequences of non-negative terms. Nonetheless, the theorem is a characterization of all classical multiplier sequences of the first kind. Indeed, since the terms of such sequences are either of the same sign or alternate in signs, and the sequences $\{-1\}_{k=0}^{\infty}$, $\{(-1)^k\}_{k=0}^{\infty}$ and $\{(-1)^{k+1}\}_{k=0}^{\infty}$ are all classical multiplier sequences (see [5, p. 341]), one achieves a complete characterization by considering merely those sequences with non-negative terms.

We close this section with the definition of the Cauchy product of two series, as this product is featured prominently in the rest of the paper.

Definition 2.6 *Let $\sum_{i=0}^{\infty} a_i x^i$ and $\sum_{j=0}^{\infty} b_j x^j$ be two power series with complex coefficients $\{a_i\}$ and $\{b_j\}$. The Cauchy product of the two power series is defined as the discrete convolution as follows:*

$$\left(\sum_{i=0}^{\infty} a_i x^i\right)\left(\sum_{j=0}^{\infty} b_j x^j\right) = \sum_{k=0}^{\infty} c_k x^k,$$

where $c_k = \sum_{\ell=0}^{k} a_\ell b_{k-\ell}$.

3 Main results

The proofs of the two main results in our paper both employ a suitably constructed linear transformation on a lattice. Here we present the preliminary results that will enable us to complete the proofs of the main results. Consider the integer lattice $\mathfrak{L}$, whose points (m, ℓ, j) are described by the inequalities

$$\begin{aligned} 0 &\le m \le n \\ 0 &\le \ell \le m \\ 0 &\le j \le n-m, \end{aligned}$$

and let M be the matrix

$$M = \begin{bmatrix} 1 & -1 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Since $\det M = 1$, $M : \mathbb{R}^3 \to \mathbb{R}^3$ is an injective linear transformation. As Lemma 3.1 shows, $M : \mathfrak{L} \to \mathfrak{L}$ is in fact a bijection.

Lemma 3.1 *Let M and $\mathfrak{L}$ be as above. Then $M : \mathfrak{L} \to \mathfrak{L}$ is a permutation.*

Proof. Since M is injective, we only need to check surjectivity. Let $(j, \ell, k) \in \mathfrak{L}$. One easily checks that

$$M^{-1}\begin{bmatrix} j \\ \ell \\ k \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} j \\ \ell \\ k \end{bmatrix} = \begin{bmatrix} k+\ell \\ k \\ j-\ell \end{bmatrix},$$

and hence $(k+\ell, k, j-\ell)$ is a point such that $M[k+\ell, k, j-\ell]^T = [j, \ell, k]^T$. What remains to verify is that $(k+\ell, k, j-\ell) \in \mathfrak{L}$. To this end, note that the restrictions

$$\begin{aligned} 0 &\le j \le n, \\ 0 &\le \ell \le j, \qquad \text{and} \\ 0 &\le k \le n-j \end{aligned}$$

imply that

$$\begin{aligned} 0 &\le k+\ell \le n, \\ 0 &\le k \le k+\ell, \qquad \text{and} \\ 0 &\le j-\ell \le n-k-\ell, \end{aligned}$$

where the last constraint comes from rearranging $k \le n-j$ to $j \le n-k$. It now follows that $(k+\ell, k, j-\ell) \in \mathfrak{L}$ if $(j, \ell, k) \in \mathfrak{L}$. The proof of the lemma is complete. □

Lemma 3.2 *Let* $(m, \ell, j) \in \mathfrak{L}$. *Then*

$$\binom{n}{m}\binom{m}{\ell}\binom{n-m}{j} = \binom{n}{m-\ell+j}\binom{m-\ell+j}{m-\ell}\binom{n-(m-\ell+j)}{\ell}$$

Proof. We compute

$$\begin{aligned} & \binom{n}{m-\ell+j}\binom{m-\ell+j}{m-\ell}\binom{n-(m-\ell+j)}{\ell} \\ = & \frac{n!(m-\ell+j)!(n-(m-\ell+j))!}{(m-\ell+j)!(n-(m-\ell+j))!(m-\ell)!\ell!j!(n-(m+j))!} \\ = & \frac{n!}{(m-\ell)!\ell!j!(n-(m+j))!} \\ = & \frac{n!m!(n-m)!}{m!(n-m)!(m-\ell)!\ell!j!(n-(m+j))!} \\ = & \binom{n}{m}\binom{m}{\ell}\binom{n-m}{j}. \end{aligned}$$

□

We need one more auxiliary lemma to prove before we can tackle the main theorems of the paper.

Lemma 3.3 *Suppose that*

$$\psi(x)=\sum_{k=0}^{\infty}\frac{\gamma_k}{k!}x^k.$$

For fixed $s,t\in(0,1)$*, let the sequence* $\{\beta_n\}_{n=0}^{\infty}$ *be defined by the relation*

$$\sum_{k=0}^{\infty}\frac{\beta_k}{k!}x^k=e^{(2-s-t)x}\psi(xt)\psi(xs).$$

Then

$$\beta_n=\sum_{m=0}^{n}\sum_{\ell=0}^{m}\sum_{j=0}^{n-m}\binom{n}{m}\binom{m}{\ell}\binom{n-m}{j}t^{\ell}s^{m-\ell}(1-t)^{j}(1-s)^{n-m-j}\gamma_\ell\gamma_{m-\ell}.$$

Proof. Suppose that $\psi(x)$ is as in the statement. We compute

$$\begin{aligned}\psi(tx)\psi(sx)&=\left(\sum_{k=0}^{\infty}\frac{\gamma_k t^k x^k}{k!}\right)\left(\sum_{j=0}^{\infty}\frac{\gamma_j s^j x^j}{j!}\right)\\&=\sum_{n=0}^{\infty}\left(\sum_{\ell=0}^{n}\frac{\gamma_\ell\ t^\ell}{\ell!}\cdot\frac{\gamma_{n-\ell}\ s^{n-\ell}}{(n-\ell)!}\right)x^n.\end{aligned}$$

It follows that

$$\begin{aligned}e^{(2-t-s)x}\psi(tx)\psi(sx)&=\left(\sum_{n=0}^{\infty}\left(\sum_{\ell=0}^{n}\binom{n}{\ell}\gamma_\ell\ \gamma_{n-\ell}t^\ell s^{n-\ell}\right)\frac{x^n}{n!}\right)\left(\sum_{k=0}^{\infty}\frac{(2-t-s)^k x^k}{k!}\right)\\&=\sum_{j=0}^{\infty}\left(\sum_{m=0}^{j}\left(\sum_{\ell=0}^{m}\binom{m}{\ell}\frac{1}{m!}\gamma_\ell\gamma_{m-\ell}t^\ell s^{n-\ell}\right)\frac{(2-t-s)^{j-m}}{(j-m)!}\right)x^j\\&=\sum_{j=0}^{\infty}\left(\sum_{m=0}^{j}\binom{j}{m}\left\{\sum_{\ell=0}^{m}\binom{m}{\ell}\gamma_\ell\ \gamma_{m-\ell}t^\ell s^{m-\ell}\right\}(2-t-s)^{j-m}\right)\frac{x^j}{j!}.\end{aligned}\tag{1}$$

Expanding and simplifying yields

$$\begin{aligned}\beta_n&=\sum_{m=0}^{n}\binom{n}{m}\left\{\sum_{\ell=0}^{m}\binom{m}{\ell}\gamma_\ell\gamma_{m-\ell}t^\ell s^{m-\ell}\right\}(2-t-s)^{n-m}\\&=\sum_{m=0}^{n}\binom{n}{m}\left\{\sum_{\ell=0}^{m}\binom{m}{\ell}\gamma_\ell\gamma_{m-\ell}t^\ell s^{m-\ell}\right\}\left(\sum_{j=0}^{n-m}\binom{n-m}{j}(1-t)^j(1-s)^{n-m-j}\right)\\&=\sum_{m=0}^{n}\sum_{\ell=0}^{m}\sum_{j=0}^{n-m}\binom{n}{m}\binom{m}{\ell}\binom{n-m}{j}t^\ell s^{m-\ell}(1-t)^j(1-s)^{n-m-j}\gamma_\ell\gamma_{m-\ell}.\end{aligned}\tag{2}$$

$\square$

We now proceed to the main results of the paper.

Definition 3.4 *Let $n \in \mathbb{Z}$ with $n \geq 0$. The n^{th} (simple) Laguerre polynomial $L_n(x)$ is given by*

$$L_n(x) = \sum_{k=0}^{n} \frac{(-1)^k}{k!} \binom{n}{k} x^k. \tag{3}$$

Theorem 3.5 *Consider the modified Bessel function of the first kind with index zero (see for example [4, p. 519]):*

$$\phi(x) = I_0(2\sqrt{x}) = \sum_{k=0}^{\infty} \frac{x^k}{k!k!}.$$

For fixed $s, t \in (0,1)$, let the sequence $\{\beta_n\}_{n=0}^{\infty}$ be defined by the relation

$$\sum_{k=0}^{\infty} \frac{\beta_k}{k!} x^k = e^{(2-s-t)x} \phi(xt)\phi(xs).$$

Then

(i) The sequence $\{\beta_n\}$ is a classical multiplier sequence;

(ii) The β_n's are given explicitly by

$$\beta_n = (1-s)^n \sum_{m=0}^{n} \left(\frac{1-t}{1-s}\right)^m \binom{n}{m} L_m\left(\frac{t}{t-1}\right) L_{n-m}\left(\frac{s}{s-1}\right)$$

where $L_n(x)$ is the n^{th} Laguerre polynomial.

Proof. (i) Definition 2.1, along with the constraints placed on t and s, imply that $e^{(2-s-t)x} \in \mathscr{L}-\mathscr{P}^+$. The fact that $\phi(x) \in \mathscr{L}-\mathscr{P}^+$ is a consequence of the fact that the sequence $\{1/k!\}_{k=0}^{\infty}$ is a classical multiplier sequence of positive terms (see for example [7, Example 44, p. 43]). Since the class $\mathscr{L}-\mathscr{P}^+$ is closed under multiplication, we conclude that $e^{(2-s-t)x}\phi(xt)\phi(xs) \in \mathscr{L}-\mathscr{P}^+$, which, by Theorem 2.4, is equivalent to $\{\beta_n\}_{n=0}^{\infty}$ being a classical multiplier sequence.

For part (ii), let $L_n(x)$ be the n^{th} simple Laguerre polynomial (cf. Definition 3.4), and write

$$\begin{aligned} C_n^{(\psi,\psi)}(t,s) &= (1-s)^n \sum_{m=0}^{n} \left(\frac{1-t}{1-s}\right)^m \binom{n}{m} L_m\left(\frac{t}{t-1}\right) L_{n-m}\left(\frac{s}{s-1}\right) \\ &= \sum_{m=0}^{n} \binom{n}{m} (1-t)^m L_m\left(\frac{t}{t-1}\right) (1-s)^{n-m} L_{n-m}\left(\frac{s}{s-1}\right). \end{aligned}$$

We now demonstrate that β_n is equal to $C_n^{(\psi,\psi)}(t,s)$. To this end, note that

$$(1-t)^m L_m\left(\frac{t}{t-1}\right) = (1-t)^m \sum_{k=0}^{m} \frac{(-1)^k}{k!}\left(\frac{t}{t-1}\right)^k \binom{m}{k}$$
$$= \sum_{k=0}^{m} \binom{m}{k} \frac{t^k(1-t)^{m-k}}{k!},$$

and

$$(1-s)^{n-m} L_{n-m}\left(\frac{s}{s-1}\right) = \sum_{j=0}^{n-m} \binom{n-m}{j} \frac{s^j(1-s)^{n-m-j}}{j!}.$$

Consequently,

$$\begin{aligned}
& (1-t)^m\, L_m\left(\frac{t}{t-1}\right)(1-s)^{n-m} L_{n-m}\left(\frac{s}{s-1}\right) \\
= & \left(\sum_{l=0}^{m} \binom{m}{\ell} \frac{1}{\ell!} t^\ell (1-t)^{m-\ell}\right)\left(\sum_{j=0}^{n-m} \binom{n-m}{j} \frac{1}{j!} s^j (1-s)^{n-m-j}\right) \\
= & \sum_{\ell=0}^{m} \sum_{j=0}^{n-m} \binom{m}{\ell}\binom{n-m}{j} t^\ell (1-t)^{m-\ell} s^j (1-s)^{n-m-j} \frac{1}{\ell! j!},
\end{aligned}$$

and

$$C_n^{(\psi,\psi)}(t,s) = \sum_{m=0}^{n} \sum_{\ell=0}^{m} \sum_{j=0}^{n-m} \binom{n}{m}\binom{m}{\ell}\binom{n-m}{j} t^\ell (1-t)^{m-\ell} s^j (1-s)^{n-m-j} \frac{1}{\ell!}\frac{1}{j!}. \tag{4}$$

We now apply the change of variables induced by M, i.e. the change $(m,\ell,j) \longrightarrow (m-\ell+j, m-\ell, \ell)$. By Lemma 3.1 the lattice $\mathfrak{L}$ is preserved under this transformation. Furthermore, the generic summand in (2) transforms as follows:

$$\begin{aligned}
& \binom{n}{m}\binom{m}{\ell}\binom{n-m}{j} t^\ell s^{m-\ell} (1-t)^j (1-s)^{n-m-j} \gamma_\ell \gamma_{m-\ell} \longrightarrow \\
& \binom{n}{m-\ell+j}\binom{m-\ell+j}{m-\ell}\binom{n-(m-\ell+j)}{\ell} t^{m-\ell} s^j (1-t)^\ell (1-s)^{n-m-j} \gamma_{m-\ell}\gamma_j.
\end{aligned}$$

We now sum over m, ℓ and j, use Lemma 3.2 and the symmetry of binomial coefficients, and substitute $\gamma_k = \frac{1}{k!}$ to arrive at the conclusion - via comparing (2) with (4) - that $\beta_n = C_n^{(\psi,\psi)}(t,s)$, as desired. $\square$

Before we can state the second claim, we give the definition of the confluent hypergeometric function and establish an auxiliary lemma.

Definition 3.6 *(see [9, Ch. 4]) The confluent hypergeometric function is given by*

$$ {}_1F_1(a;b;z) = 1 + \frac{a}{b}z + \frac{a(a+1)}{b(b+1)}\frac{z^2}{2!} + \ldots = \sum_{k=0}^{\infty} \frac{(a)_k}{(b)_k}\frac{z^k}{k!} \tag{5} $$

where $(a)_k := a(a+1)\cdots(a+k-1)$ is the rising factorial. If a and b are integers, $a \le 0$, and either $b > 0$ or $b < a$, then the hypergeometric series yields a polynomial with a finite number of terms. If $b \in \mathbb{Z}$ and $b \le 0$, then ${}_1F_1(a;b;z)$ is undefined.

Lemma 3.7 *For $k \in \mathbb{Z}$ such that $k \ge 0$,*

$$ \frac{4^{-k}}{\left(\frac{1}{2}\right)_k} = \frac{k!}{(2k)!}. $$

Proof. Note that $(x)_k = \frac{\Gamma(x+k)}{\Gamma(x)}$, $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$, and $\Gamma\left(\frac{1}{2}+k\right) = \frac{(2k)!}{4^k k!}\sqrt{\pi}$. With these identities in hand and with the choice $x = 1/2$ we obtain

$$ \begin{aligned} \frac{4^{-k}}{\left(\frac{1}{2}\right)_k} &= \frac{4^{-k}}{\frac{\Gamma\left(\frac{1}{2}+k\right)}{\Gamma\left(\frac{1}{2}\right)}} \\ &= \frac{4^{-k}\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}+k\right)} \\ &= \frac{4^{-k}\sqrt{\pi}}{\frac{(2k)!}{4^k k!}\sqrt{\pi}} \\ &= \frac{4^{-k}4^k k!}{(2k)!} \\ &= \frac{k!}{(2k)!}. \end{aligned} $$

□

We are now ready to formulate the second result.

Theorem 3.8 *Consider the modified hyperbolic cosine function $\phi(x) = \sum_{k=0}^{\infty} \frac{x^k}{(2k)!}$. Let the sequence $\{\beta_k\}_{k=0}^{\infty}$ be defined by*

$$ \sum_{k=0}^{\infty} \frac{\beta_k}{k!} x^k = e^{(2-s-t)x}\phi(xt)\phi(xs), $$

where $s, t \in (0,1)$.
Then

(i) The sequence $\{\beta_n\}$ is a classical multiplier sequence;

(ii) The β_n are given by the expression

$$(1-s)^n \sum_{m=0}^{n} \binom{n}{m} \left(\frac{1-t}{1-s}\right)^m {}_1F_1\left[-m; \frac{1}{2}; \frac{t}{4(t-1)}\right] {}_1F_1\left[-(n-m); \frac{1}{2}; \frac{s}{4(s-1)}\right].$$

Proof. In the proof of Theorem 3.5 we already argued that $e^{(2-s-t)x} \in \mathscr{L}-\mathscr{P}^+$. We note that $\phi(x)$ is in $\mathscr{L}-\mathscr{P}^+$, since

$$\phi(x) = \cosh(\sqrt{x}) = \sum_{k=0}^{\infty} \frac{x^k}{(2k)!} = \prod_{k=0}^{\infty} \left(1 + \frac{x}{\left(\pi k + \frac{\pi}{2}\right)^2}\right)$$

is a factorization of the form required by Definition 2.1. Since $\mathscr{L}-\mathscr{P}^+$ is closed under multiplication, we obtain that $\{\beta_n\}_{n=0}^{\infty}$ is a classical multiplier sequence, which was our first claim.
(ii) By Lemma 3.3, we have the following representation for β_n:

$$\beta_n = \sum_{m=0}^{n} \sum_{\ell=0}^{m} \sum_{j=0}^{n-m} \binom{n}{m}\binom{m}{\ell}\binom{n-m}{j} t^\ell s^{m-\ell}(1-t)^j(1-s)^{n-m-j}\gamma_\ell\gamma_{m-\ell}.$$

Using the same change of variables and arguments identical to those in the proof of Theorem 3.5, we obtain the 'transformed' representation

$$\beta_n = \sum_{m=0}^{n} \sum_{\ell=0}^{m} \sum_{j=0}^{n-m} \binom{n}{m}\binom{m}{\ell}\binom{n-m}{j} (1-t)^\ell t^{m-\ell} s^j (1-s)^{n-m-j}\gamma_j\gamma_{m-\ell}. \tag{6}$$

We now calculate

$$\begin{aligned}
&\sum_{m=0}^{n} \sum_{\ell=0}^{m} \sum_{j=0}^{n-m} \binom{n}{m}\binom{m}{\ell}\binom{n-m}{j} (1-t)^\ell t^{m-\ell} s^j (1-s)^{n-m-j}\gamma_j\gamma_{m-\ell} \\
=\ & (1-s)^n \sum_{m=0}^{n} \sum_{\ell=0}^{m} \binom{n}{m}\binom{m}{\ell}(1-t)^\ell t^{m-\ell}\gamma_{m-\ell} \sum_{j=0}^{n-m}\binom{n-m}{j} s^j(1-s)^{-m-j}\gamma_j \\
\overset{*}{=}\ & (1-s)^n \sum_{m=0}^{n} \binom{n}{m}(1-s)^{-m} \sum_{\ell=0}^{m}\binom{m}{\ell}(1-t)^{m-\ell}t^\ell\gamma_\ell \sum_{j=0}^{n-m}\binom{n-m}{j} s^j(1-s)^{-j}\gamma_j \\
\overset{**}{=}\ & (1-s)^n \sum_{m=0}^{n} \binom{n}{m}\left(\frac{1-t}{1-s}\right)^m \sum_{\ell=0}^{m}\binom{m}{\ell}\frac{\ell!}{(2\ell)!}\left(\frac{t}{1-t}\right)^\ell \sum_{j=0}^{n-m}\binom{n-m}{j}\frac{j!}{(2j)!}\left(\frac{s}{1-s}\right)^j \\
\overset{***}{=}\ & (1-s)^n \sum_{m=0}^{n} \binom{n}{m}\left(\frac{1-t}{1-s}\right)^m {}_1F_1\left[-m; \frac{1}{2}; \frac{t}{4(t-1)}\right] {}_1F_1\left[-(n-m); \frac{1}{2}; \frac{s}{4(s-1)}\right],
\end{aligned}$$

where the starred equality follows from reversing the middle sum, the double starred equality follows from substituting $\gamma_k = \frac{k!}{(2k)!}$, and the triple starred equality follows from

Lemma 3.7 and some algebraic manipulation. This completes the proof of Theorem 3.8. □

Remark: The expression in Equation (6) is the one we get if we do the Cauchy products

$$\left(e^{(1-t)x}\phi(xt)\right)\left(e^{(1-s)x}\phi(xs)\right),$$

which in turn can be expressed in terms of the Jensen polynomials associated to $\cosh(x)$. As the authors of [3] show, the coefficients β_n can be expressed in terms of the Jensen polynomials. The evaluation this way leads to the same conclusions with equally tedious and lengthy calculations.

4 Classical multiplier sequences and the $C_k^{\varphi,\Phi}(t,s)$ representation

The following definition and theorem appear in [3] (Definition 6 and Theorem 19, p. 1384).

Definition 4.1 *Let $\gamma_k \in \mathbb{R}$ for $k = 0, 1, 2, \ldots$. We say that the sequence $\{\gamma_k\}_{k=0}^{\infty}$ has a C_k-representation, if there exist functions $\varphi, \Phi \in \mathscr{L}-\mathscr{P}^+$ (not necessarily distinct) and $s, t \in \mathbb{R}$ such that $\gamma_k = C_k^{\varphi,\Phi}(t,s)$ for all $k \in \mathbb{N}_0$, where $C_k^{\varphi,\Phi}(t,s)$ is as defined by the generating relation*

$$e^{((1-t)+(1-s))x}\varphi(xt)\Phi(xs) = \sum_{k=0}^{\infty} \frac{C_k^{\varphi,\Phi}(t,s)}{k!}x^k.$$

Theorem 4.2 *(1) Every polynomially interpolated multiplier sequence of non-negative terms can be written as a sequence $\{C_k^{\varphi,\Phi}(t,s)\}_{k=0}^{\infty}$ for some functions $\varphi, \Phi \in \mathscr{L}-\mathscr{P}^+$ and $(t,s) \in [0,1]\times[0,1]$. The choice of φ, Φ, t and s in this representation need not be unique.*
(2) Every geometric multiplier sequence (i.e., a sequence of the form $\{r^k\}_{k=0}^{\infty}$, $r \in \mathbb{R}$) has a C_k-representation.

Some of the details of the proof of the theorem were left somewhat vague in [3]. In addition, there are aspects of this theorem we wanted to explore further. More explicitly, we seek answers to the following questions:

(i) If $\{\gamma_k\}$ has a $C_k^{\varphi,\Phi}$-representation, must it have one with $\varphi = \Phi$? Similarly, if $\{\gamma_k\}$ has a $C_k^{\varphi,\varphi}$-representation, must it have one with $\varphi \neq \Phi$?

(ii) Demonstrate explicitly that the $C_k^{\varphi,\Phi}$-representation is not unique. What if we require $\varphi = \Phi$. Is the C_k representation unique then?

We begin our discussion by substantiating the first claim in Theorem 4.2. To this end, recall that a sequence $\{\gamma_k\}_{k=0}^{\infty}$ of non-negative terms is a CMS if and only if $\varphi = \sum_{k=0}^{\infty}\frac{\gamma_k}{k!}x^k \in \mathscr{L}-\mathscr{P}^+$. If we set $\varphi(x) = \sum_{k=0}^{\infty}\frac{\gamma_k}{k!}x^k$, $\Phi(x) \equiv 1$, and $t = s = 1$. Then,

$$e^{((1-t)+(1-s))x}\varphi(xt)\Phi(xs) = e^{0\cdot x}\varphi(x)\cdot 1 = \sum_{k=0}^{\infty}\frac{\gamma_k}{k!}x^k.$$

Therefore, every CMS of non-negative terms has a $C_k^{\varphi,\Phi}(t,s)$ representation. Since most of what follows exploits the form of $\widehat{p}(x)$ in the relation $\sum p(k)/k!x^k = \widehat{p}(x)e^x$, we present here a result regarding $\widehat{p}(x)$ (the statement of Theorem 4.3 appears without proof in [3, p. 1385].)

Theorem 4.3 *For any $p \in \mathbb{R}[x]$,*

$$\sum_{k=0}^{\infty} \frac{p(k)}{k!} x^k = \widehat{p}(x)e^x,$$

where $\widehat{p}(x) = a_0 + \sum_{j=1}^{n} \left(\sum_{k=j}^{n} a_k S_2(k,j)\right) x^j$, and $S_2(k,j)$ denote the Stirling numbers of the second kind.

In order to be able to prove the theorem, we need the following preliminary lemma. Since the result is known, we omit the proof here, but remark that one can obtain it by induction and using identities of the Stirling numbers.

Lemma 4.4 *[2, Theorem 2] For any $n \in \mathbb{N}$,*

$$(xD)^n[e^x] = \left(\sum_{j=1}^{n} S_2(n,j)x^j\right) e^x,$$

where $S_2(k,j)$ denotes the Stirling numbers of the second kind ([10, Ch. 7]).

Proof. (of Theorem 4.3). We will prove the theorem by induction on the degree of p. When $n = 1$, we may write $p(x) = a_1 x + a_0$. With this formulation, we calculate

$$\begin{aligned}
\sum_{k=0}^{\infty} \frac{p(k)}{k!} x^k &= (a_1 xD + a_0)[e^x] \\
&= (a_1 x + a_0)e^x \\
&= \left(a_0 + \sum_{j=1}^{1} \left(\sum_{k=1}^{1} a_k S_2(k,j)\right) x^j\right) e^x.
\end{aligned}$$

We now proceed with the induction step.
Note that if the degree of p is n and $p(x) = \sum_{k=0}^{n} a_k x^k$, then

$$\begin{aligned}
\widehat{p}(x)e^x &= \sum_{k=0}^{\infty} \frac{p(k)}{k!} x^k \\
&= \sum_{k=0}^{\infty} \frac{\sum_{j=0}^{n-1} a_j k^j + a_n k^n}{k!} x^k
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=0}^{\infty} \frac{\sum_{j=0}^{n-1} a_j k^j}{k!} x^k + \sum_{k=0}^{\infty} \frac{a_n k^n}{k!} x^k \\
&= \widehat{q}(x)[e^x] + a_n (xD)^n [e^x] \\
&= \left(\widehat{q}(x) + a_n \left(\sum_{j=1}^{n} S_2(n,j) x^j \right) \right) e^x \\
&= \left(a_0 + \sum_{j=1}^{n-1} \left(\sum_{k=j}^{n-1} a_k S_2(k,j) \right) x^j + \left(\sum_{j=1}^{n} a_n S_2(n,j) x^j \right) \right) e^x \\
&= \left(a_0 + \sum_{j=1}^{n} \left(\sum_{k=j}^{n} a_k S_2(k,j) \right) x^j \right) e^x,
\end{aligned}
$$

and the proof is complete. □

We now answer question (i) in the negative. That is, we show that not every CMS of non-negative terms has a $C_k^{\varphi,\Phi}(t,s)$ representation with $\varphi = \Phi$.

Lemma 4.5 *There exist classical multiplier sequences of non-negative terms that admit no $C_k^{\varphi,\Phi}(t,s)$ representation with $\varphi = \Phi$.*

Proof. Suppose $\{\gamma_k\}_{k=0}^{\infty} = \{p(k)\}_{k=0}^{\infty}$ is a classical multiplier sequence, where p is a monic quadratic polynomial. Then,

$$
\sum_{k=0}^{\infty} \frac{\gamma_k}{k!} x^k \in \mathscr{L} - \mathscr{P}^{+}
$$

and

$$
\begin{aligned}
\sum_{k=0}^{\infty} \frac{\gamma_k}{k!} x^k &= \sum_{k=0}^{\infty} \frac{p(k)}{k!} x^k \\
&= \widehat{p}(x) e^x \\
&= (x + r_1)(x + r_2) e^x, \qquad (7)
\end{aligned}
$$

where $r_1, r_2 \geq 0$. Assuming that a $C_k^{\varphi,\varphi}(t,s)$ representation exists, φ would have to be linear, say $\varphi(x) = mx + b$. Consequently,

$$
\varphi(tx) = mtx + b = x + r_1
$$

and

$$
\varphi(sx) = msx + b = x + r_2.
$$

By looking at the coefficient of x in the exponential factor in (7), we conclude that $(1-t) + (1-s) = 1$, and hence $t + s = 1$. The above equations imply that $mt = ms = 1$, and thus $t = s = \frac{1}{m} = \frac{1}{2}$. However, the above system is still inconsistent, if $r_1 \neq r_2$. We conclude, that if $p(x) = x^2 + a_1 x + a_0$ and $(a_1 + 1)^2 - 4a_0 > 0$, then $\{p(k)\}_{k=0}^{\infty}$ does *not* have a $C_k^{\varphi,\Phi}(t,s)$ representation with $\varphi = \Phi$. □

The converse of question (i) is settled in the following lemma.

Lemma 4.6 *If a classical multiplier sequence* $\{\gamma_k\}_{k=0}^{\infty}$ *has a* $C_k^{\varphi,\varphi}$*-representation, then it also has one with* $\varphi \neq \Phi$*.*

Proof. The idea here is that if $\varphi(x)$ is a polynomial of degree n with only real non-positive zeros, then for all $r \neq 0$,

$$\begin{aligned}
\varphi(tx) &= c(tx)^m \prod_{k=1}^{n-m}\left(1+\frac{tx}{x_k}\right) \\
&= r^n c\left(\frac{t}{r}x\right)^m \prod_{k=1}^{n-m}\left(\frac{1}{r}+\frac{tx}{rx_k}\right) \\
&= r^n \bar{\varphi}(tx).
\end{aligned}$$

Thus if we set $\Phi(x) = r^n\varphi(x)$, then $\bar{\varphi}(xt)\Phi(xs) = \varphi(xt)\varphi(xs)$, but $\bar{\varphi} \neq \varphi$, and $\Phi \neq \varphi$. $\square$

An argument similar to the proof of Lemma 4.6 establishes that the functions in the $C_k^{\varphi,\Phi}(t,s)$ of a classical multiplier sequence are not unique.

Lemma 4.7 *The functions in the* $C_k^{\varphi,\Phi}(t,s)$ *representation of a classical multiplier sequence are not necessarily unique.*

Proof. Suppose that p is a cubic polynomial so that $\{p(k)\}_{k=0}^{\infty}$ is a classical multiplier sequence. We write

$$\sum_{k=0}^{\infty}\frac{p(k)}{k!}x^k = x(x+r_1)(x+r_2)(x+r_3)e^x, \qquad r_1,r_2,r_3 \in (-\infty,0].$$

Note that

$$x(x+r_1)(x+r_2)(x+r_3)e^x = 16\left(\frac{x}{2}\right)\left(\frac{x}{2}+\frac{r_1}{2}\right)\left(\frac{x}{2}+\frac{r_2}{2}\right)\left(\frac{x}{2}+\frac{r_3}{2}\right)e^{(2-1/2-1/2)x}.$$

We define

$$\begin{aligned}
\varphi_1(x) &= 4x \\
\Phi_1(x) &= 4\left(x+\frac{r_1}{2}\right)\left(x+\frac{r_2}{2}\right)\left(x+\frac{r_3}{2}\right) \\
\varphi_2(x) &= 2x\left(x+\frac{r_1}{2}\right) \\
\Phi_2(x) &= 8\left(x+\frac{r_2}{2}\right)\left(x+\frac{r_3}{2}\right).
\end{aligned}$$

It is clear that $\varphi_1 \neq \Phi_1$, $\varphi_2 \neq \Phi_2$, and $\varphi_1 \neq \varphi_2$, $\Phi_1 \neq \Phi_2$, although

$$\begin{aligned} e^{((1-1/2)+(1-1/2))x}\varphi_1\left(\frac{x}{2}\right)\Phi_1\left(\frac{x}{2}\right) &= \sum_{k=0}^{\infty} \frac{C_k^{\varphi_1,\Phi_1}\left(\frac{1}{2},\frac{1}{2}\right)}{k!}x^k \\ &= \sum_{k=0}^{\infty} \frac{p(k)}{k!}x^k \\ &= \sum_{k=0}^{\infty} \frac{C_k^{\varphi_2,\Phi_2}\left(\frac{1}{2},\frac{1}{2}\right)}{k!}x^k \\ &= e^{((1-1/2)+(1-1/2))x}\varphi_2\left(\frac{x}{2}\right)\Phi_2\left(\frac{x}{2}\right). \end{aligned}$$

We thus conclude that the functions a $C_k^{\varphi,\Phi}(t,s)$ representation of a classical multiplier sequence are in fact not unique. □

The second part of question (ii) is addressed next.

Lemma 4.8 *There exist classical multiplier sequences with two distinct $C_k^{\Phi,\Phi}(t,s)$ representations.*

Proof. The key point here is that the pair (t,s) is part of the representation. With this note in hand, a simple example demonstrates the claim. Suppose we let $\varphi \equiv 1$ and $\Phi \equiv 1$. Then,

$$e^{((1-1/3)+(1-2/3))x}\varphi\left(\frac{1x}{3}\right)\varphi\left(\frac{2x}{3}\right) = e^{((1-1/5)+(1-4/5))x}\Phi\left(\frac{x}{5}\right)\Phi\left(\frac{4x}{5}\right)$$

and hence

$$\left\{C_k^{\varphi,\varphi}\left(\frac{1}{3},\frac{2}{3}\right)\right\}_{k=0}^{\infty} = \left\{C_k^{\Phi,\Phi}\left(\frac{1}{5},\frac{4}{5}\right)\right\}_{k=0}^{\infty}.$$

□

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