The Coloring Graph of Complete Graphs

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Abstract - We study the coloring graph of the family of complete graphs and we prove that $\mathcal{C}_n(K_t)$ is regular, transitive, and connected when $n > t$. Also, we study whether $\mathcal{C}_n(K_t)$ is distance transitive or strongly regular, and find its diameter.

Keywords : coloring graph; transitive graphs; complete graphs

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1 Background

This article pertains to graphs and to objects that are related to certain well-studied properties of graphs. Hence, in this section, we provide notation and background on objects, definitions, and results that will be necessary throughout this work. Most of these concepts are well-known but we still direct the curious reader to [4] for more on graph theory, and for a deeper look of some concepts that we may not fully cover here.

We start with some definitions.

A graph G is an ordered pair $G = (V, E)$ where V is the set of vertices of the graph and E is the set of edges of the graph, which connect pairs of vertices of the graph. All the graphs in this article are simple, i.e they have no loops or multiple edges. A graph coloring is an assignment of labels, "colors", to elements of a graph subject to certain constraints.

Now we define the main object of this work.

Definition 1.1 *Let $G = (V, E)$ be a graph. Let $n \in \mathbb{N}$. A proper n-coloring of G is a function $f : V \to \{1, 2, \ldots, n\}$ so that no two adjacent vertices have the same image under f.*

Colorings of graphs have been extensively studied. A quick search in MathSciNet for graph colorings yields 8,000+ hits. A new spin on this concept has recently gained some attention: the coloring graphs of a graph.

Definition 1.2 *Let G be a graph and $n \in \mathbb{N}$. The n-coloring graph of G, denoted $\mathcal{C}_n(G)$, is the graph with vertex-set, the set of all proper n-colorings of G and defining edges only between n-colorings that differ at exactly one vertex of G. The vertices in $\mathcal{C}_n(G)$ are denoted in boldface (e.g. $\mathbf{v}$).*

There are many open questions on coloring graphs. For example, it is not known what graphs can be represented as the coloring graph of some other graph; recent efforts have focused on looking at forbidden subgraphs for coloring graphs.

Theorem 1.3 (Beier, Fierson, Haas, Russell, and Shavo [3]) *C_3, C_4 and C_6 are the only cycles that are coloring graphs. In fact, if a graph has C_5 as an induced subgraph then that graph is not a coloring graph. Also, every tree is the induced subgraph of some coloring graph.*

Also, [3] provides a table with all 27 (non-isomorphic) graphs with up to 12 vertices that are coloring graphs. The reader is invited to also read [1], where more results on forbidden subgraphs are obtained.

Under a different perspective, researchers are also interested in obtaining conditions for a coloring graph to be connected. The following two results tell us about connectivity of $C_k(G)$ depending on the graph's chromatic number, $\chi(G)$ (the minimum number of colors needed to properly color G), or the graph's coloring number, $col(G)$ (the least κ for which there exists a vertex-ordering of G in which each vertex has fewer than κ neighbors that are earlier in the ordering. See [8]).

Theorem 1.4 (Cereceda, van den Heuvel, and Johnson [6]) *If $\chi(G) = k \in \{2, 3\}$, then $C_k(G)$ is not connected. On the other hand, for $k \geq 4$ there are examples of connected and of not connected coloring graphs with $\chi(G) = k$.*

Theorem 1.5 (Dyer, Flaxman, Frieze, and Vigoda [7]) *For any graph G and integer $k \geq col(G) + 1$, $C_k(G)$ is connected.*

Also, in [2], the authors study whether connectivity implies biconnectivity (a graph on n vertices is biconnected if 2 is the size of the smallest subset of vertices such that the graph becomes disconnected if these vertices are deleted). They obtain an affirmative result when $n = 3$.

Theorem 1.6 (Bhakta, Buckner, Farquhar, Kamat, Krehbiel, and Russell [2]) *Let G be an arbitrary graph. If $\mathcal{C}_3(G)$ is connected, then $\mathcal{C}_3(G)$ is biconnected.*

Here are some definitions specific to this paper.

A vertex w is a neighbor of v if there is an edge connecting them, this edge will be denoted vw. The degree of a vertex v, denoted $deg(v)$, is the number of neighbors of v. A regular graph of degree k is a graph in which every vertex has degree k. A path on n vertices, denoted P_n, is a graph in which $n-2$ vertices have degree two and the other two vertices have degree one. A complete graph, denoted K_t, is a graph in which all of its t vertices are connected with one another. The diameter of G is the greatest distance between any pair of vertices of G. The girth of a graph is the shortest graph cycle in the graph. A regular graph, G, with ν vertices and degree k is said to be strongly regular if there are also integers λ and μ such that: (1) every two adjacent vertices have exactly λ common neighbors and (2) every two non-adjacent vertices have exactly μ common neighbors. We will say that a strongly regular graph has parameters (ν, k, λ, μ). The following remark about strongly regular graphs is a well-known result (e.g. see [5], page 12).

Remark 1.7 The parameters (ν, k, λ, μ) of a strongly regular graph must obey the relation

$$(\nu - k - 1)\mu = k(k - \lambda - 1)$$

An automorphism of a graph $G = (V, E)$ is a map $\phi : V \to V$ such that ϕ is bijective and for which $v_1 v_2 \in E$ if and only if $f(v_1)f(v_2) \in E$. The set of all automorphisms of G is denoted $Aut(G)$, and it is a group under composition. G is (vertex) transitive if and only if, given $v_1, v_2 \in V$ there is $\phi \in Aut(G)$ so that $\phi(v_1) = v_2$. G is distance transitive if, given $v, w, x, y \in V(G)$ so that $d(v, w) = d(x, y)$ then there is $\phi \in Aut(G)$ so that $\phi(v) = x$ and $\phi(w) = y$. Also, for a fixed $k \in \mathbb{N}$, a graph G is k-distance transitive if given $v, w, x, y \in V(G)$ so that $d(v, w) = k = d(x, y)$ then there is $\phi \in Aut(G)$ so that $\phi(v) = x$ and $\phi(w) = y$. Clearly, a graph is distance transitive if it is k-distance transitive for all $k \in \mathbb{N}$.

Also, there are not many families of graphs for which their coloring graph has been completely understood; that is where our work fits in. In this article, we investigate the coloring graphs of complete graphs by looking into their structure (e.g. regularity), and by looking at their automorphism groups and related properties (e.g. transitivity).

2 The Coloring Graph of Complete Graphs

Very few families of coloring graphs have been obtained by analyzing a given family of graphs. We next look at the coloring graphs of complete graphs.

Definition 2.1 *A graph is said to be complete on t vertices, denoted K_t, if $|V| = t$ and it is regular of degree $(t-1)$.*

We will use the following notation for the vertices of $\mathcal{C}_n(K_t)$. We fix an ordering of the vertices of K_t, $v_1, v_2, \ldots, v_t$. Let f be an n-coloring of K_t and $c_i = f(v_1)$. Note that, since K_t is complete, the 'colors' $c_1, c_2, \ldots, c_t$ are all distinct elements of $\{1, 2, \ldots, n\}$, thus $n \geq t$. An n-coloring of K_t will be denoted by $\langle c_1, c_2, \ldots, c_t \rangle$.

Theorem 2.2 *Let $n \geq t \geq 1$. Then, $\mathcal{C}_n(K_t)$ is regular with degree $(n-t)t$. Moreover, the $(n-t)t$ neighbors of any given vertex of $\mathcal{C}_n(K_t)$ partition into t copies of K_{n-t}.*

Proof. Note that $\mathcal{C}_n(K_1)$ is a complete graph on n vertices. Hence, the claim is true for $t = 1$. We will prove the claim for the case $t > 1$ by induction on $k = n - t$.

When $k = 0$ (and $t > 1$), it is easy to see that $\mathcal{C}_t(K_t)$ has $t!$ disjoint vertices (each having degree 0, naturally). Hence, the claim holds in this base case.

Now assume that the result holds for all graphs $\mathcal{C}_m(K_s)$, where $0 \leq m - s < k$, for some $k > 0$, and for $s > 1$. Consider the graph $\mathcal{C}_n(K_t)$, where $n - t = k > 0$ and $t > 1$, and notice that it has $C_{n-1}(K_t)$ as a subgraph. Note that $(n-1) - t = k \geq 0$.

For any given $(n-1)$-coloring of K_t, say $\langle c_1, c_2, \ldots, c_t \rangle$, where $c_i \in \{1, \ldots, n-1\}$, we can create t new neighbors of it in $\mathcal{C}_n(K_t) \setminus \mathcal{C}_{n-1}(K_t)$ by considering

$$\langle n, c_2, \ldots, c_t \rangle,\ \langle c_1, n, c_3, \ldots, c_t \rangle, \cdots,\ \langle c_1, c_2, \ldots, n, c_t \rangle,\ \langle c_1, c_2, \ldots, n \rangle$$

Doing this for all vertices of $\mathcal{C}_{n-1}(K_t)$ yields all vertices of $\mathcal{C}_n(K_t)$, and thus the degree of a vertex in $\mathcal{C}_n(K_t)$ is

$$((n-1)-t)t + t = (n-t)t.$$

This proves the first claim.

The second claim follows by noticing that given a vertex $\langle c_1, c_2, \ldots, c_t \rangle$ and a 'component', $i \in \{1, 2, \ldots, t\}$, then there are exactly $n-t$ possible values that can be used to replace c_i. All these colorings will determine a K_{n-t} in $\mathcal{C}_n(K_t)$. □

We get a couple of easy corollaries.

Corollary 2.3 *Given a vertex* $\mathbf{v}$ *of* $\mathcal{C}_n(K_t)$*, the neighborhood of* $\mathbf{v}$ *(including* $\mathbf{v}$*) in* $\mathcal{C}_n(K_t)$ *is* t *copies of* K_{n-t+1}*, all sharing* $\mathbf{v}$ *and disjoint otherwise.*

Corollary 2.4 *Let* $n - t > 1$*. Then, the girth of* $\mathcal{C}_n(K_t)$ *is equal to* 3*.*

Now that we have established regularity of $\mathcal{C}_n(K_t)$, we look at whether it is strongly regular.

Lemma 2.5 *Let* $n > t \geq 1$*. The number of common neighbors shared by two adjacent vertices in* $\mathcal{C}_n(K_t)$ *is* $\lambda = n - t - 1$*.*

Proof. This follows from Corollary 2.3. Two adjacent vertices, $\mathbf{v}$ and $\mathbf{w}$, in $\mathcal{C}_n(K_t)$ must be in one of the K_{n-t+1} found based at, WLOG, $\mathbf{v}$. It follows that there are $n-t-1$ common neighbors of $\mathbf{v}$ and $\mathbf{w}$. □

Although we were able to find λ in Lemma 2.5, $\mathcal{C}_n(K_t)$ is not strongly regular.

Theorem 2.6 *The coloring graph* $\mathcal{C}_n(K_t)$ *where* t *is fixed and* $n > t \geq 3$ *is not strongly regular.*

Proof. We proceed by contradiction. If $\mathcal{C}_n(K_t)$ were strongly regular, then it should have parameters (v, k, λ, μ) satisfying the relation given in Remark 1.7. Easy computations and Lemma 2.5 yield

$$v = \frac{n!}{(n-t)!} \qquad k = t(n-t) \qquad \lambda = n - t - 1$$

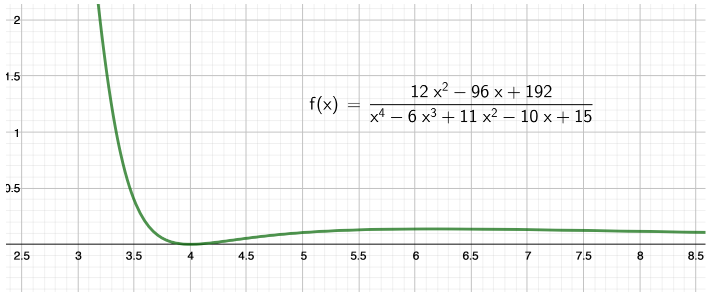
With these values, the equation in Remark 1.7 is

$$\left(\frac{n!}{(n-t)!} - t(n-t) - 1\right)\mu = t(n-t)(t(n-t) - (n-t-1) - 1)$$

which implies

$$\mu = \frac{t(n-t)(t(n-t) - n + t)}{\frac{n!}{(n-t)!} - t(n-t) - 1} \tag{1}$$

For the cases $t = 3, 4, 5, 6$, we plot (using GeoGebra) μ as a function of n and see that the values for μ are less than one for $n > t$. For example, the graph of μ for $t = 4$ follows.



This contradicts the fact that μ is a non-negative integer.

In order to reach a similar contradiction for $n > t > 6$, we will bound Equation (1) above by an expression that is still less than one. We first use that t and n are positive to bound the numerator of Equation (1):

$$\mu = \frac{t(n-t)(t(n-t)-n+t)}{\frac{n!}{(n-t)!} - t(n-t) - 1} \leq \frac{tn(tn+t)}{\frac{n!}{(n-t)!} - t(n-t) - 1} = \frac{t^2 n(n+1)}{\frac{n!}{(n-t)!} - t(n-t) - 1}$$

Now we look at the denominator of the bound we just obtained and use that $t^2 - 1 > 0$ to get

$$\frac{t^2 n(n+1)}{\frac{n!}{(n-t)!} - t(n-t) - 1} \leq \frac{t^2 n(n+1)}{\frac{n!}{(n-t)!} - tn} = \frac{t^2(n+1)}{\frac{(n-1)!}{(n-t)!} - t}$$

Since $n > t$ we get

$$\frac{t^2(n+1)}{\frac{(n-1)!}{(n-t)!} - t} \leq \frac{n^2(n+1)}{\frac{(n-1)!}{(n-t)!} - n}$$

Finally, we use that $t > 6$ to get

$$\frac{n^2(n+1)}{\frac{(n-1)!}{(n-t)!} - n} \leq \frac{n^2(n+1)}{(n-1)(n-2)(n-3)(n-4)(n-5) - n}$$

Overall, we get

$$\mu \leq \frac{n^2(n+1)}{(n-1)(n-2)(n-3)(n-4)(n-5) - n} \tag{2}$$

Next, as we did for the cases $t = 3, 4, 5, 6$, we plot the expression on the right-hand side of Inequality (2) (using GeoGebra) to get that for any $n > 6$ the function has all values being less than one. This contradicts μ being a non-negative integer. Thus, $\mathcal{C}_n(K_t)$ is not strongly regular. □

We close this section with a quick look at the connectedness of this graph. Theorem 1.4 tells us that $\mathcal{C}_t(K_t)$ is not connected for $t = 2, 3$. For all other cases, we get the following result.

Theorem 2.7 *$\mathcal{C}_n(K_t)$ is connected, for $n > t$. $\mathcal{C}_t(K_t)$ is not connected, for all $t > 1$.*

Proof. Since all vertex orderings of K_t are equivalent, and all the vertices of K_t have $t-1$ neighbors, we get that $col(K_t) = t$. Hence, Theorem 1.5 implies that if $n \geq t+1$, $C_n(K_t)$ is connected.

It is easy to see that $\mathcal{C}_t(K_t)$ is $t!$ disconnected vertices, and thus not connected, for all $t > 1$. □

In order to find the diameter of $\mathcal{C}_n(K_t)$, we need the following definition.

Definition 2.8 *A derangement of a set with t elements is an element in S_t, such that no element appears in its original position. That is, a derangement is a permutation that has no fixed points.*

Theorem 2.9 *Let $n > t \geq 2$. The diameter of $\mathcal{C}_n(K_t)$ is $\frac{3t}{2}$, if t is even, or $4 + \frac{3(t-3)}{2}$, if t is odd.*

Proof. First notice that, since $n > t \geq 2$, Theorem 2.7 tells us that $\mathcal{C}_n(K_t)$ is connected.

We will prove the following claim: for $2 \leq t < n$, the distance between two vertices of $\mathcal{C}_n(K_t)$ is at most $\frac{3t}{2}$, if t is even, or $4 + \frac{3(t-3)}{2}$, if t is odd. Moreover, for every t, there is a pair of vertices for which the bound is reached; this pair of vertices of $\mathcal{C}_n(K_t)$ are derangements of the same t colors.

Let us first consider the case $t = 2$. It is easy to see that, given two vertices of $\mathcal{C}_n(K_2)$, the distance between them is at most three, which is reached in the case when the vertices are of the form $\langle c_1, c_2 \rangle$ and $\langle c_2, c_1 \rangle$ and the path connecting them is

$$\langle c_1, c_2 \rangle \to \langle x, c_2 \rangle \to \langle x, c_1 \rangle \to \langle c_2, c_1 \rangle$$

where $c_1 \neq x \neq c_2$.

We will prove that the claim holds for all other cases by induction on n. For $n = 3$, we only have to look at $\mathcal{C}_3(K_2)$, which falls into the previously-discussed case $t = 2$. Next we fix $n > 3$ and assume that the claim is true for all $\mathcal{C}_m(K_t)$, where $2 \leq t < m < n$.

Consider the graph $\mathcal{C}_n(K_t)$, where $2 < t < n$, and the vertices $\mathbf{v} = \langle c_1, c_2, \ldots, c_t \rangle$ and $\mathbf{w} = \langle d_1, d_2, \ldots, d_t \rangle$.

Case 1: Assume that $\mathbf{v}$ and $\mathbf{w}$ coincide in at least one vertex/component. Assume WLOG that $c_t = d_t$, and consider $\mathbf{v'} = \langle c_1, c_2, \ldots, c_{t-1} \rangle$ and $\mathbf{w'} = \langle d_1, d_2, \ldots, d_{t-1} \rangle$ be two vertices of $\mathcal{C}_{n-1}(K_{t-1})$. By induction, the distance between $\mathbf{v'}$ and $\mathbf{w'}$ is at most $\frac{3(t-1)}{2}$, if t is odd, or $4 + \frac{3(t-4)}{2}$, if t is even. Note that

$$\frac{3(t-1)}{2} = \frac{3 \cdot 2}{2} + \frac{3(t-3)}{2} < 4 + \frac{3(t-3)}{2} \tag{3}$$

if t is odd, and

$$4 + \frac{3(t-4)}{2} = \frac{8 + 3(t-4)}{2} < \frac{3t}{2} \tag{4}$$

if t is even. In any case, the distance between $\mathbf{v}$ and $\mathbf{w}$ is at most $\frac{3t}{2}$, if t is even, or $4 + \frac{3(t-3)}{2}$, if t is odd.

Case 2: Assume that $\mathbf{v}$ and $\mathbf{w}$ do not coincide in any vertices and that $\mathbf{w}$ is not a derangement of $\mathbf{v}$. It follows that one vertex of $\mathbf{v}$ is not a vertex of $\mathbf{w}$. WLOG, say it is c_1. Note that $\mathbf{w}$ is adjacent in $\mathcal{C}_n(K_t)$ with $\langle c_1, d_2, \ldots, d_t \rangle$, which by Case 1 is at distance at most $\frac{3(t-1)}{2}$, if t is odd, or $4+\frac{3(t-4)}{2}$, if t is even, from $\mathbf{v}$. The result follows from Equations (3) and (4), now featuring non-strict inequalities because of the distance one step from $\mathbf{w}$ to $\langle c_1, d_2, \ldots, d_t \rangle$.

Case 3: Assume that $\mathbf{w}$ is a derangement of $\mathbf{v}$. Let τ be the derangement acting on $\mathbf{v}$ that yields $\mathbf{w}$ and let, WLOG, $(c_1, c_2, \ldots, c_k)$ be a cycle in the (disjoint) cycle decomposition of τ. Note that $k \neq 1, t-1$. Consider the path in $\mathcal{C}_n(K_t)$, where $x \neq c_i$ for all i,

$$\langle c_1, c_2, c_3, \ldots, c_t \rangle \to \langle x, c_2, c_3, \ldots, c_t \rangle \to \langle x, c_1, c_3, \ldots, c_t \rangle \to \langle x, c_1, c_2, \ldots, c_t \rangle$$

and so on, until we get $\mathbf{u} = \langle c_k, c_1, c_2, \ldots, c_t \rangle$. This path has length $k+1$. Note that $\mathbf{v}$ and $\mathbf{u}$ share k components and that the vertices $\mathbf{v}\prime = \langle c_{k+1}, c_{k+2}, \ldots, c_t \rangle$ and $\mathbf{w}\prime = \langle d_{k+1}, d_{k+2}, \ldots, d_t \rangle$ are still connected by a derangement. By induction, the distance between $\mathbf{v}\prime$ and $\mathbf{w}\prime$ is at most

$$\max\left\{\frac{3(t-k)}{2}, 4+\frac{3(t-k-3)}{2}\right\} = \frac{3(t-k)}{2}$$

Thus the length of the path connecting $\mathbf{v}$ and $\mathbf{w}$ is at most

$$\frac{3(t-k)}{2} + k + 1 = \frac{3(t-k)+2(k+1)}{2} = \frac{3(t-3)}{2} + \frac{11-k}{2} \leq 4 + \frac{3(t-3)}{2}$$

if $k \geq 3$. For when $k=2$, assume WLOG that $d_1 = c_2$ and $d_2 = c_1$. We first get the path, where $x \neq c_i$ for all i,

$$\langle c_1, c_2, c_3, \ldots, c_t \rangle \to \langle x, c_2, c_3, \ldots, c_t \rangle \to \langle x, c_1, c_3, \ldots, c_t \rangle \to \langle c_2, c_1, c_3, \ldots, c_t \rangle$$

which has length 3. Since $\langle c_3, \ldots, c_t \rangle$ is a derangement of $\langle d_3, \ldots, d_t \rangle$, then the distance between these two vertices is at most $\frac{3(t-2)}{2}$, if t is even, or $4+\frac{3(t-5)}{2}$, if t is odd. Hence, the distance between $\mathbf{v}$ and $\mathbf{w}$ is at most

$$\frac{3(t-2)}{2} + 3 = \frac{3t}{2}$$

if t is even, and

$$4 + \frac{3(t-5)}{2} + 3 = 4 + \frac{3(t-3)}{2}$$

if t is odd. Note that, by induction, there is a pair of vertices (one a derangement of the other) for which the bounds are reached. $\square$

3 Automorphisms of $\mathcal{C}_n(K_t)$

Now we look at the automorphism group of $\mathcal{C}_n(K_t)$ to prove that it acts transitively on the vertices of $\mathcal{C}_n(K_t)$. We start with two theorems that will tell us about large families of automorphisms of $\mathcal{C}_n(K_t)$.

Lemma 3.1 *Let $\tau \in S_t$. The function $g_\tau : V(\mathcal{C}_n(K_t)) \to V(\mathcal{C}_n(K_t))$, defined by*

$$g_\tau\left(\left\langle c_1, c_2, \ldots, c_t\right\rangle\right) = \left\langle c_{\tau(1)}, c_{\tau(2)}, \ldots, c_{\tau(t)}\right\rangle$$

is an automorphism of $\mathcal{C}_n(K_t)$.

Proof. Since τ is injective, we get that $g_\tau(\mathbf{v}) \in V(\mathcal{C}_n(K_t))$, for all $\mathbf{v} \in V(\mathcal{C}_n(K_t))$. Hence, g_τ is well defined. In order to check that g_τ is injective, we pick two vertices of $\mathcal{C}_n(K_t)$: $\mathbf{v} = \left\langle c_1, c_2, \ldots, c_t\right\rangle$ and $\mathbf{w} = \left\langle c_1\prime, c_2\prime, \ldots, c_t\prime\right\rangle$.

$$\begin{aligned} g_\tau(\mathbf{v}) &= g_\tau(\mathbf{w}) \\ g_\tau\left(\left\langle c_1, c_2, \ldots, c_t\right\rangle\right) &= g_\tau\left(\left\langle c_1\prime, c_2\prime, \ldots, c_t\prime\right\rangle\right) \\ \left\langle c_{\tau(1)}, c_{\tau(2)}, \ldots, c_{\tau(t)}\right\rangle &= \left\langle c_{\tau(1)}\prime, c_{\tau(2)}\prime, \ldots, c_{\tau(t)}\prime\right\rangle \end{aligned}$$

From this we can see that $c_{\tau(i)} = c_{\tau(i)}\prime$, for all $i = 1, 2, \ldots, t$. Since τ is bijective, we get that $c_i = c_i\prime$, for all $i = 1, 2, \ldots, t$, and thus $\mathbf{v} = \mathbf{w}$ and g_τ is one-to-one.

Now we consider $\mathbf{v} = \left\langle c_1\prime, c_2\prime, \ldots c_t\prime\right\rangle \in V(C_n(K_t))$ and let

$$\mathbf{w} = \left\langle c_{\sigma^{-1}(1)}\prime, c_{\sigma^{-1}(2)}\prime, \ldots, c_{\sigma^{-1}(t)}\prime\right\rangle \in V(C_n(K_t))$$

Clearly, $g_\tau(\mathbf{v}) = \mathbf{w}$. So, g_τ is onto.

Next consider two adjacent vertices in $\mathcal{C}_n(K_t)$: WLOG, let $\mathbf{v} = \left\langle c_1, c_2, \ldots, c_t\right\rangle$ and $\mathbf{w} = \left\langle d_1, d_2, \ldots, d_t\right\rangle$, where $c_i = d_i$, for all $i = 1, \ldots, t-1$ and $c_t \neq d_t$. Assume $\tau(t) = j$, for some $j \in \{1, \ldots, t\}$. It follows that

$$g_\tau(\mathbf{v}) = \left\langle c_{\tau(1)}, c_{\tau(2)}, \ldots, c_{\tau(t)}\right\rangle \qquad g_\tau(\mathbf{w}) = \left\langle d_{\tau(1)}, d_{\tau(2)}, \ldots, d_{\tau(t)}\right\rangle$$

coincide at all 'components' except for the ith components, where we find c_t and d_t, which are distinct. Hence, $g_\tau(\mathbf{v})$ and $g_\tau(\mathbf{w})$ are adjacent. Since g_τ is bijective and preserves incidence, g_τ is an automorphism of $C_n(K_t)$. □

Using similar ideas, we can prove the following theorem as well.

Lemma 3.2 *Let $\sigma \in S_n$. The function $f_\sigma : V(\mathcal{C}_n(K_t)) \to V(\mathcal{C}_n(K_t))$, defined by*

$$f_\sigma\left(\left\langle c_1, c_2, \ldots, c_t\right\rangle\right) = \left\langle \sigma(c_1), \sigma(c_2), \ldots, \sigma(c_t)\right\rangle$$

is an automorphism of $\mathcal{C}_n(K_t)$.

Lemmas 3.1 and 3.2 give us that both S_n and S_t are subgroups of $Aut(\mathcal{C}_n(K_t))$. Our next theorem extends our understanding of $Aut(\mathcal{C}_n(K_t))$ even more.

Remark 3.3 Lemma 3.2 is true for all graphs and not just complete graphs. This result is mentioned in [3], page 3, though it is not proven, and it is also used in [9] to define the *isomorphic coloring graph*.

Theorem 3.4 *Let $\tau \in S_t$ and $\sigma \in S_n$. Then, the automorphisms f_σ and g_τ commute. Hence, there is a subgroup of $Aut(\mathcal{C}_n(K_t))$ that is isomorphic to $S_n \times S_t$.*

Proof. Let $< c_1, \dots, c_t >$ be a vertex of $\mathcal{C}_n(K_t)$, and let $\tau \in S_t$ and $\sigma \in S_n$. Let $d_i = \sigma^{-1}(c_i)$. We get

$$\begin{aligned} f_\sigma(g_\tau(f_\sigma^{-1} < c_1, \dots, c_t >)) &= f_\sigma(g_\tau < d_1, \dots, d_t >) \\ &= f_\sigma < \sigma^{-1}(c_{\tau(1)}), \dots, \sigma^{-1}(c_{\tau(t)}) > \\ &=< \sigma(\sigma^{-1}(c_{\tau(1)})), \dots, \sigma(\sigma^{-1}(c_{\tau(t)})) > \\ &=< c_{\tau(1)}, \dots, c_{\tau(t)} > \\ &= g_\tau < c_1, \dots, c_t > \end{aligned}$$

Hence, $f_\sigma g_\tau f_\sigma^{-1} = g_\tau$, and thus f_σ and g_τ commute.

Since the group of all functions of the form f_σ is isomorphic to S_n and the group of all functions of the form g_τ is isomorphic to S_t, $Aut(\mathcal{C}_n(K_t))$ contains a subgroup isomorphic to $S_n \times S_t$. □

Next we make use of the automorphisms we just found to prove that $\mathcal{C}_n(K_t)$ is vertex transitive.

Theorem 3.5 *$\mathcal{C}_n(K_t)$ is vertex transitive.*

Proof. Let $\mathbf{v} = \langle c_1, c_2, \dots, c_t \rangle$ and $\mathbf{w} = \langle d_1, d_2, \dots, d_t \rangle$ be two vertices in $V(\mathcal{C}_n(K_t))$. Since the sets $\{c_1, c_2, \dots, c_t\}$ and $\{d_1, d_2, \dots, d_t\}$ consist of t distinct elements each, we can define a bijection $\overline{\sigma} : \{c_1, c_2, \dots, c_t\} \to \{d_1, d_2, \dots, d_t\}$ by $\overline{\sigma}(c_i) = d_i$, for all $i = 1, \dots, t$. Since the domain and range of $\overline{\sigma}$ are subsets of $\{1, \dots, n\}$, we can extend $\overline{\sigma}$ to $\sigma \in S_n$. It follows that $f_\sigma(\mathbf{v}) = \mathbf{w}$. Hence, $\mathcal{C}_n(K_t)$ is vertex transitive. □

We next look at whether $\mathcal{C}_n(K_t)$ is d-distance transitive, for small values of d.

Theorem 3.6 *$\mathcal{C}_n(K_t)$ is 1-distance transitive.*

Proof. We will prove that $Aut(\mathcal{C}_n(K_t))$ acts transitively on the set of vertices at distance 1 from $\mathbf{v}$, for a fixed vertex $\mathbf{v}$. This will be enough because if $\mathbf{w}$ is any other vertex and $\mathbf{w}_1$ and $\mathbf{w}_2$ are vertices at distance one from $\mathbf{w}$, then we consider $\phi \in Aut(\mathcal{C}_n(K_t))$ be so that $\phi(\mathbf{w}) = \mathbf{v}$, and let $\phi(\mathbf{w}_1) = \mathbf{v}_1$ and $\phi(\mathbf{w}_2) = \mathbf{v}_2$, which are both at distance one from $\mathbf{v}$. Since we are assuming that $Aut(\mathcal{C}_n(K_t))$ acts transitively on the set of vertices at distance 1 from $\mathbf{v}$, there is $\varphi \in Aut(\mathcal{C}_n(K_t))$ so that $\varphi(\mathbf{v}_1) = \mathbf{v}_2$. It follows that $\phi \circ \varphi \circ \phi^{-1}$ maps $\mathbf{w}_1$ into $\mathbf{w}_2$.

Let us now prove that $Aut(\mathcal{C}_n(K_t))$ acts transitively on the set of vertices at distance 1 from $\mathbf{v}$, for a fixed vertex $\mathbf{v}$. WLOG, we assume $\mathbf{v} = \langle 1, 2, \dots, t \rangle$. The set S of vertices at distance 1 from $\mathbf{v}$ is

$$S = \{\langle c_1, c_2, \dots, c_t \rangle;\ c_i = i, \text{ for all but exactly one } i\}$$

Consider, WLOG, the two elements of S given by

$$\mathbf{u} = \langle x_1, 2, 3, \dots, t\rangle \qquad \mathbf{w} = \langle 1, x_2, 3, \dots, t\rangle$$

where $x_1, x_2 \notin \{1, 2, \dots, t\}$. Let $\tau = (1\ 2\ \cdots\ t) \in S_t$ and $\sigma = (1\ 2\ \cdots\ t)(x_1\ x_2) \in S_n$. Note that $f_\sigma(g_\tau(\mathbf{v})) = \mathbf{v}$ and that

$$f_\sigma(g_\tau(\mathbf{u})) = f_\sigma\langle t, x_1, 2, 3, \dots, t-1\rangle = \langle 1, x_2, 3, 4, \dots, t\rangle = \mathbf{w}$$

It follows that $C_n(K_t)$ is 1-distance transitive. □

Remark 3.7 The argument used to prove Theorem 3.6 may be extended to 2-distance transitivity. However, it cannot be extended to 3-distance transitivity. An example follows.

Let $t \geq 3$ and $n \geq t+1$. Consider $\mathbf{v} = \langle 1, 2, \dots, t\rangle$ and let $x_1 \notin \{1, \dots, t\}$. Then,

$$\mathbf{u} = \langle 2, 1, 3, 4, \dots, t\rangle \qquad \mathbf{w} = \langle x_1, 1, 2, 4, \dots, t\rangle$$

are at distance 3 from $\mathbf{v}$. If there were $\sigma \in S_n$ and $\tau \in S_t$ so that $f_\sigma(g_\tau(\mathbf{v})) = \mathbf{v}$, then σ could not map elements in $\{1, 2, \dots, t\}$ to elements in $\{t+1, \dots, n\}$, and vice-versa. So, it would be impossible for $f_\sigma \circ g_\tau$ to also map $\mathbf{u}$ to $\mathbf{w}$, as no matter what τ is, g_τ would only permute the 'coordinates' of the colorings and thus we would not be able to obtain the x_1 we need for $\mathbf{w}$. Hence, we cannot decide whether $\mathcal{C}_n(K_t)$ is 3-distance transitive from what we currently know about $Aut(\mathcal{C}_n(K_t))$ (Theorem 3.4).

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