Prime Labelings of Snake Graphs

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Abstract - A prime labeling of a graph G with n vertices is a labeling of the vertices with distinct integers from the set $\{1, 2, \ldots, n\}$ such that the labels of any two adjacent vertices are relatively prime. In this paper, we introduce a snake graph, the fused union of identical cycles, and define a consecutive snake prime labeling for this new family of graphs. We characterize some snake graphs that have a consecutive snake prime labeling and then consider a variation of this labeling.

Keywords : graphs; graph labelings; prime labelings; cycles

Mathematics Subject Classification (2010) **:** 05C78

1 Introduction

A *graph* $G = (V, E)$ is a set of n vertices connected by a set of edges E. A *graph labeling* is an assignment of integers to the vertices, edges, or both, such that some constraint is satisfied. In the late 1960s, Rosa [6] introduced several variations of graph labelings that later became known as variations of *graceful labelings.* Graceful labelings assign values to each vertex so that the absolute difference between adjacent vertices is unique along each edge of the graph. The study of graceful labelings has led to developments in coding theory, communication network designs, optimization problems, and number theory problems [3].

While graceful labelings were studied across many different classes of graphs, a variety of other labeling methods, such as *near graceful, harmonious, cordial, magic,* and *anti-magic* labelings began to develop. In 1980, Entringer developed another popular labeling, the *prime labeling* [2]. Prime labelings were first studied by Tout, Dabboucy, and Howalla [8] in 1982. A graph has a prime labeling if the vertices of G can be labeled distinctly with the integers $\{1, 2, \ldots, n\}$, where n is the number of vertices, such that any two adjacent vertex labels are relatively prime. A graph is said to be prime if it has a prime labeling.

Like graceful labelings, prime labelings have been studied extensively across a large number of graph families. Deretsky, Lee, and Mitchem [2] proved that all cycle graphs C_n are prime. Berliner, et al. [1] examined the prime labelings of ladder graphs and complete bipartite graphs. Schluchter, et al. [7] analyzed prime labelings of generalized Petersen

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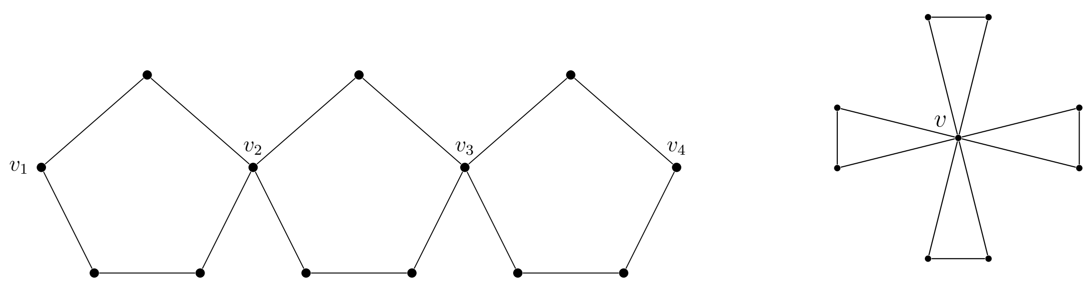
graphs and were able to conclude all generalized Petersen graphs $P(n,k)$ are prime for even $n < 50$ and odd k. Perhaps one of the most popular graph families to study for prime labelings is trees. In the early 1980s, Entringer [4] conjectured that all trees have a prime labeling. Several classes of trees have been proven prime including paths, stars, spiders, and many other specific variations [4]. In 2012, Haxell, Pikhurko, and Taraz [5] showed that all large order trees were prime, essentially confirming the conjecture.

In this paper, we explore prime labelings of snake graphs, $C^m_{k,q}$. Snakes are the *fusion* of m cycle graphs, C_k, where q is the minimal number of edges between each point of fusion, as defined in Definition 2.2. In Section 2, we define the components for snake graphs and a *cyclic snake labeling* before exploring the results of applying this labeling function to snake graphs. We describe how snake labelings differ as the parameters m, k, and q vary. We examine another labeling, similar to the cyclic snake labeling, in Section 3. Finally, we offer direction for further work in Section 4.

2 Cyclic Snakes

We begin by introducing some terminology regarding prime labelings and snake graphs. Most notably, the idea of a snake graph has been adapted to more closely resemble work done on cycle graphs, resulting in the use of C, instead of S, in the notation.

Definition 2.1 *A snake graph $C^m_{k,q}$ is the fusion of m k-cycles, C_k, such that, for $2 \leq i \leq m$, a shared vertex called the vertebrae, denoted v_i, results from the fusion where a minimal path of length q joins v_{i-1} and v_i.*



(a) $C^3_{5,2}$ with the four vertebrae, v_1, v_2, v_3, and v_4 labeled.

(b) An example of the fan graph, or $C^4_{3,0}$. Note here that $v = v_1 = v_2 = v_3 = v_4 = v_5$, showing that vertebrae need not be unique.

Figure 1: Two examples of snake graphs. In (a) there are $m = 3$ 5-cycles, c_5 connected by a minimal path of length $q = 2$. In (b), we have $m = 4$ 3-cycles connected by a minimal path of length $q = 0$.

Two observations from this definition are important. First, if $q = 0$ then there is a minimal path of length 0 between all vertebrae. This results in one shared vertex between every cycle, forming a fan graph. An example is given in Figure 1b. Second, this definition implies that all cycles of the form C_k are snakes where $m = 1$ and $1 \le q \le \lfloor \frac{k}{2} \rfloor$.

Definition 2.2 *The points of fusion, denoted v_i, are the internal vertebrae of the snake, $2 \le i \le m$. Additionally, the vertebra v_1 is a vertex in the first cycle that has a path of q edges between it and v_2 while the vertebra v_{m+1} is a vertex in the m^{th} cycle that has a path of q edges between it and v_m.*

As seen in Figure 1b, there is a minimal path of length $q = 0$ between each cycle, resulting in $v_1 = v_2 = v_3 = v_4 = v_5$. Thus there is only one vertebra for this snake graph.

Definition 2.3 *A minimal path of length qm from v_1 to v_{m+1} is called the spine. The vertices along the spine not shared between two cycles are referred to as s^i_ℓ, where i refers to the cycle which the spine vertex belongs and ℓ refers to the distance between s^i_ℓ and v_i. For all snakes, $1 \le i \le m$ and $1 \le \ell \le q-1$.*

Definition 2.4 *The path of $(k-q)m$ edges from v_1 to v_{m+1} that does not contain any vertices on the spine is called the belly of the snake graph. The vertices along the belly not shared between two cycles are referred to as b^i_j, where i refers to the cycle which the belly vertex belongs and j specifies the distance between b^i_j and v_i. For all snakes, $1 \le i \le m$ and $1 \le j \le k-q-1$.*

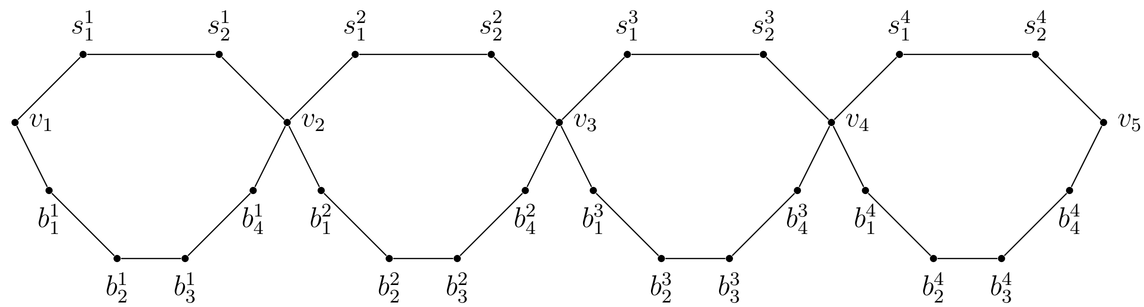


Figure 2: The snake graph $C^4_{8,3}$. The spine of the snake, the minimal path of length qm between vertices v_1 and v_5, can be observed along the top of the figure. The belly of the snake, the path of length $(k-q)m$ between vertices v_1 and v_5, can be seen along the bottom of the figure.

We now develop an algorithm for a cyclic snake labeling using the above definitions of the vertebrae, spine, and belly vertices. A cyclic snake labeling begins with the vertebra v_1 and labels it 1. From v_1, travel along the belly, labeling each vertex consecutively until reaching the final belly vertex of that cycle, b^1_{k-q-1}. After labeling the final belly vertex of the first cycle, move to the spine of the snake to continue labeling consecutively from

s_1^1 to s_{q-1}^1. Finish the cycle by labeling v_2 as k. Repeat the same steps when labeling the rest of the cycles on the snake. This leads to the following formal definition.

Definition 2.5 *A cyclic snake labeling is defined by the following bijective mapping* $f(x) : V\left(C_{k,q}^m\right) \to \{1, 2, \ldots, n\}$. *For any vertex* x *in* $C_{k,q}^m$ *such that* $1 \leq i \leq m+1$, $1 \leq j \leq k-q-1$, *and* $1 \leq \ell \leq q-1$

$$f(x) = \begin{cases} ik - i - k + 2 & \textit{if } x = v_i \\ ik - i + 1 - q + \ell & \textit{if } x = s_\ell^i \\ ik - i - k + j + 2 & \textit{if } x = b_j^i. \end{cases}$$

If f *results in a prime graph, then we say* f *is a cyclic snake prime labeling.*

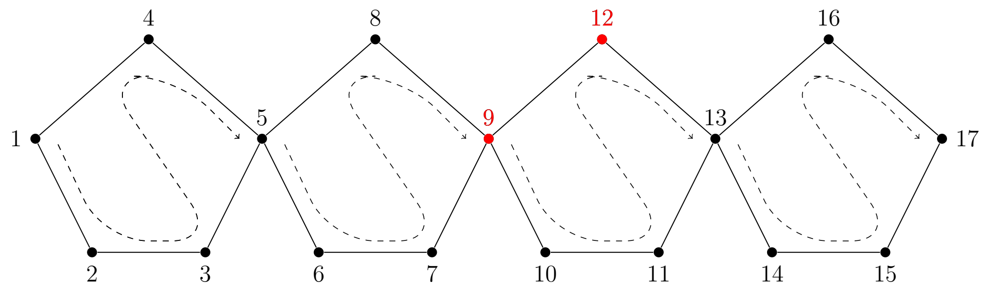


Figure 3: Snake $C_{5,2}^4$ labeled with a cyclic snake labeling as in Definition 2.5. By following the dashed arrows left to right, we see that the labels themselves "snake" around each cycle. We label the leftmost vertebra, the first belly left to right, the first spine left to right, and then the rightmost vertebra, the starting vertex of the next cycle. Because v_3 and s_1^3, labeled 9 and 12, share a common factor, the cyclic snake labeling is not a prime labeling for this graph.

It is important to note that in a cyclic snake labeling, $f(b_1^i) = f(v_i) + 1$ and $f(s_{q-1}^i) = f(s_{i+1}) - 1$. Since the labels of these pairs of vertices are consecutive, they are relatively prime. The only adjacent vertices not necessarily receiving consecutive labels are $f(v_i)$ and $f(s_1^i)$ as well as $f(v_{i+1})$ and $f(b_{k-q-1}^i)$.

In some cases, f results in a prime labeling, but in others, it does not, as illustrated in Figure 3. We analyze these cases and present modifications that can result in a prime labeling for specific cases of snake graphs. We also analyze cases of the cyclic snake labeling that are never prime. However, this is not to say that a cyclic snake labeling is the only possible prime labeling for a snake $C_{k,q}^m$. For example, the cyclic snake labeling of $C_{5,2}^4$ is not prime. However, a prime labeling of $C_{5,2}^4$ exists and is shown in Figure 4. This paper is focused on the primeness of snakes with a cyclic snake labeling but not the general primeness of snakes.

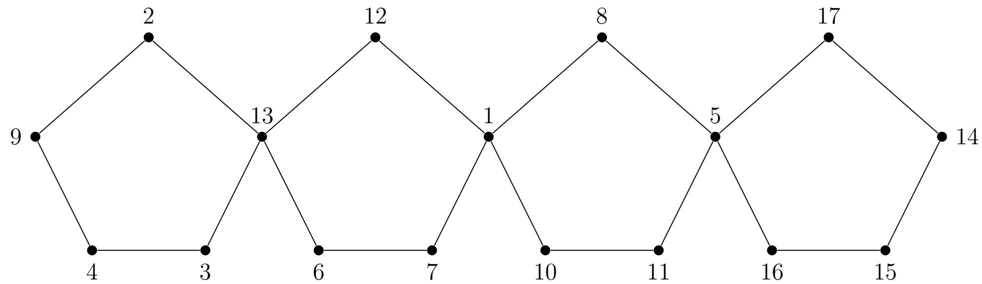


Figure 4: Snake $C^4_{5,2}$ labeled with a non cyclic snake labeling. This labeling is prime, but because it does not follow our cyclic snake labeling it is not included in our results below.

2.1 Preliminary Results

We first look at general results for snake graphs and prime labelings. These results are useful in narrowing our considerations as we examine specified values for q and how the q value affects the relationships between vertex labels.

Our first proposition allows us to quickly determine whether two adjacent vertex labels are relatively prime when the labels are both one greater than a multiple of their difference.

Proposition 2.6 *Let $\{x_i\}_{i\geq 0}$ be the sequence defined by $x_i := 1 + id$, $d \in \mathbb{Z}$. Then x_i and x_{i+1} are relatively prime for any i.*

Proof. Let $k \in \mathbb{Z}^+$ such that k divides x_i and x_{i+1}. That is, for $i, d \in \mathbb{Z}^+$, k must divide both $id+1$ and $d(i+1)+1$. By the division algorithm, $id+1 = kq_1$ and $d(i+1)+1 = kq_2$ for some quotients $q_1, q_2 \in \mathbb{N}$. Observe that $d = d(i+1)+1-(id+1) = kq_2 - kq_1 = k(q_2 - q_1)$. Therefore, k must divide d.

Now, let $(q_2 - q_1) = r$ such that $d = kr$. Because k divides both $id+1$ and $d(i+1)+1$, then k must divide $id + 1 = ikr + 1$. Thus, $k = 1$, and x_i and x_{i+1} must be relatively prime. $\square$

Similarly, the following proposition is helpful in quickly determining whether the label of a spine vertex s^i_1 is relatively prime to the adjacent vertebra vertex, v_i.

Proposition 2.7 *For all m, if k is odd, then the vertebrae labels, $f(v_i)$, of $C^m_{k,q}$ are also odd $(1 \leq i \leq m + 1)$.*

Proof. We first show that the final vertebra v_{m+1} has an odd label by showing that $f(v_{m+1})$ is odd for all m. Then it follows from Definition 2.5 that $f(v_{m+1}) = (m + 1)k - (m + 1) - k + 2 = k(m) - m + 1 = m(k - 1) + 1$. Because k is odd, $k - 1$ is even. Thus, $f(v_{m+1})$ is odd.

Now observe from Definition 2.5 that $f(v_i) - f(v_{i-1}) = k - 1$. Therefore, if $f(v_{m+1})$ is odd, all vertebrae labels $f(v_i)$ will also be odd. $\square$

The following proposition serves as a starting point for our first theorem concerning $C^m_{k,q}$.

Proposition 2.8 *If k and q are both even, then a cyclic snake labeling of $C^m_{k,q}$ is never prime.*

Proof. Let $G = C^m_{k,q}$ be a snake graph where k and q are both even. Label $C^m_{k,q}$ using the cyclic snake prime labeling from Definition 2.5. Observe that $f(v_2) = k$ and is therefore even. Moreover, the belly vertex in the first cycle adjacent to v_2, b^1_{k-q-1}, will also be labeled with an even integer since $f(b^1_{k-q-1}) = k - q$. Because k and q are both even, $k - q$ is also even. Thus, $f(v_2)$ and $f(b^1_{k-q-1})$ are both even and are not relatively prime. Therefore, the cyclic snake labeling of $C^m_{k,q}$, with k and q even, is never prime. $\square$

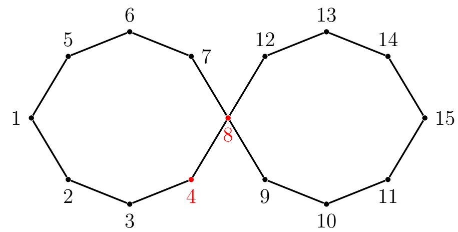


Figure 5: An example of the cyclic snake labeling for $C^2_{8,4}$. Because $k = 8$ and $q = 4$, this is not a cyclic snake prime labeling, as proven in Proposition 2.8.

Figure 5 illustrates Proposition 2.8. This proposition allows us to eliminate a large number of snake graphs when considering the primality of a cyclic snake labeling. If q is even, we only need to consider those snakes composed of odd cycles. We can extend the ideas of Proposition 2.8, to obtain the following statement on the prime labeling of a snake graph.

Theorem 2.9 *If q divides k, then the cyclic snake labeling of $C^m_{k,q}$ is never prime.*

Proof. Let $G = C^m_{k,q}$ such that q divides k, and let $i = 1$. By the definition of a cyclic snake labeling, $f(v_2) = k = qj$ for some positive integer j. The belly vertebra adjacent to v_2, b^1_{k-q-1}, has label $f(b^1_{k-q-1}) = k - q$, by Definition 2.5. Substituting, we observe $f(b^1_{k-q-1}) = qj - q = q(j-1)$. Thus, $f(v_2)$ and $f(b^1_{k-q-1})$ share a common factor of q, and $C^m_{k,q}$ is not prime under the cyclic snake labeling. $\square$

The snake graph in Figure 5, also illustrates Proposition 2.9 because $k = 8$ and $q = 4$. We see that $q = 4$ divides both the label of v_2 and the last belly vertex on the first cycle, b^1_3.

The propositions above help us to more quickly examine and determine the primality of a snake graph's cyclic snake labeling. Moving forward, we consider more specific snake graphs and when a cyclic snake labeling produces a prime labeling. In the following sections we focus on specific values of q.

2.2 Snakes with q = 1

The spinal vertices of snake graphs of the form $C_{k,1}^m$ are the vertebrae, as shown in Figure 6. We are able to show that all snake graphs of this type have a cyclic snake prime labeling.

Theorem 2.10 *Every snake graph $C_{k,1}^m$ is prime with a cyclic snake labeling.*

Proof. First observe that the Hamiltonian path containing the vertebrae and belly vertices in $C_{k,1}^m$ is always prime since the labels of this path are consecutive integers. Because of this, we need only check that consecutive vertebrae v_i and v_{i+1} are relatively prime for $2 \le i \le m$.

Since the vertebrae labels begin with 1 and increase by $k-1$ to each successive v_i, by Proposition 2.6, all vertebrae labels are relatively prime. Therefore the graph labeling of $C_{k,1}^m$ is prime for all values of m and k. □

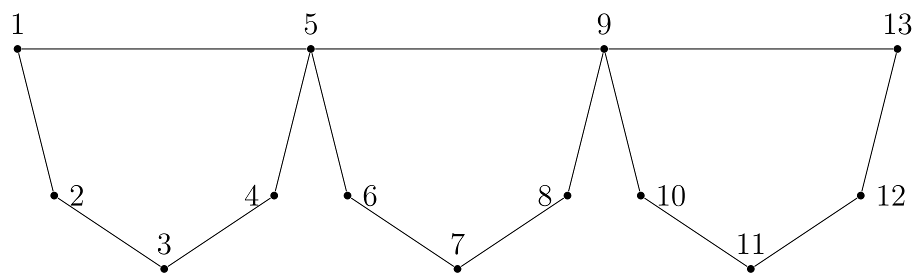


Figure 6: An example of a snake graph $C_{5,1}^3$. This snake has a cyclic snake prime labeling.

2.3 Snakes with q = 2

To aid us in our exploration of snakes with larger q values, we begin with a few remarks on snake graphs when $q > 1$.

Remark 2.11 Let G be a snake with $q > 1$. Then G has two pairs of non-consecutive labels in the i^{th} cycle, $1 \le i \le m$. The first pair is the vertebra vertex v_i and the first spinal vertex s_1^i. The labels $f(v_i)$ and $f(s_1^i)$ will always be distance $k-q$ apart. The second pair of non-consecutive vertex labels in the i^{th} cycle is the vertebra v_{i+1} and the final belly vertex b_{k-q-1}^i. The assigned labels, $f(v_{i+1})$ and $f(b_{k-q-1}^i)$, will always be distance q apart.

An example illustrating the labeling pattern shown in Remark 2.11 is shown in Figure 7.

Lemma 2.12 *If the absolute difference of integers x and y is a prime p, then the greatest common divisor of x and y is either p or 1. If p does not divide one of x or y, then x and y are relatively prime.*

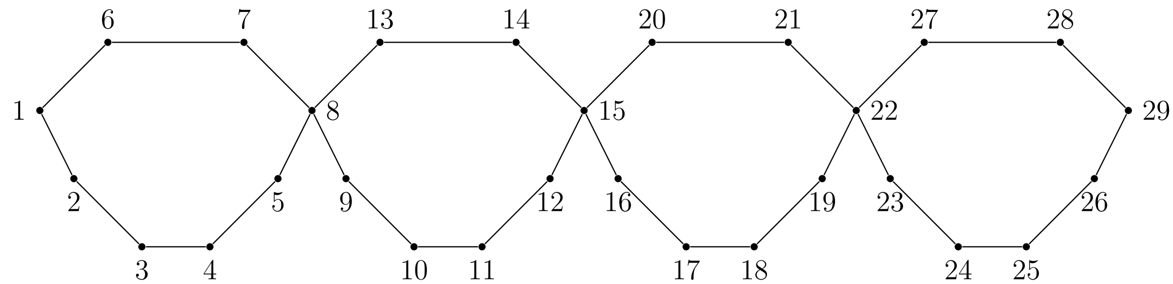


Figure 7: Snake graph $C^4_{8,3}$ with labels to illustrate Remark 3.1.

Proof. Let x and y have an absolute difference of a prime p, and let their prime factorizations be $p_1^{k_1}p_2^{k_2}\cdots p_n^{k_n}$ and $q_1^{j_1}q_2^{j_2}\cdots q_m^{j_m}$ respectively. Let d be the greatest common factor of x and y. Then

$$\begin{aligned} x-y &= (p_1^{k_1}p_2^{k_2}\cdots p_n^{k_n}) - (q_1^{j^1}q_2^{j_2}\cdots q_m^{j_m}) \\ p &= d(c_x - c_y). \end{aligned}$$

Then either $d=1$ or p. If neither x nor y is a multiple of p, then the greatest common divisor cannot be p, and thus x and y are relatively prime. □

Using Lemma 2.12 and Remark 2.11, we can quickly show when the two sets of nonconsecutive labels are relatively prime.

Theorem 2.13 *For every snake graph $C^m_{k,2}$ where k is odd, the cyclic snake labeling is prime if and only if m is less than the smallest prime factor of $k-2$.*

Proof. Let p be the smallest prime factor of $k-2$. We will first show that if $m \geq p$, then the consecutive snake labeling is not prime. By Remark 2.11, we need only consider the nonconsecutive labels, $f(v_i)$ and $f(s_1^i)$ as well as $f(v_{i+1})$ and $f(b^i_{k-q-1})$ for $1 \leq i \leq m$.

Let us first look at the relationship between $f(v_p)$ and $f(s_1^p)$. Because $q=2$, $f(s_1^p) = f(v_{p+1}) - 1 = f(v_p) + (k-2)$. Since p divides $k-2$, $k-2 = pc$ for some $c \in \mathbb{Z}$. We now consider the p^{th} cycle of $C^m_{k,2}$.

$$\begin{aligned} f(v_p) &= p(k-1) + pc = p(k-1+c) \\ f(s_1^p) &= p(k-1) \end{aligned}$$

Therefore $f(v_p)$ and $f(s_1^p)$ share a common factor of p and are not relatively prime. Thus, if $m \geq p$ the the labeling is not a cyclic snake prime labeling.

Now we show that when $m < p$, all adjacent vertex labels are relatively prime. Again, by Remark 2.11, we need only consider the nonconsecutive labels, $f(v_i)$ and $f(s_1^i)$ as well as $f(v_{i+1})$ and $f(b^i_{k-q-1})$, for $1 \leq i \leq m$.

Assume some integer d is the smallest common factor for $f(v_i)$ and $f(s_1^i)$. Then there exist q_1 and q_2 such that $dq_1 = f(v_i)$ and $dq_2 = f(s_1^i)$. By Remark 2.11, $f(s_1^i) - f(v_i) = k-2$, implying d divides $k-2$. Now consider $f(v_i) = ik - i - k + 2 = i(k-1) - (k-2)$. Because d divides $k-2$, it cannot also divide $k-1$ and must divide i. Consequently, $d \leq i < p$, and $d = 1$. Therefore, $f(v_i)$ and $f(s_1^i)$ are also relatively prime.

The relationship between $f(v_i)$ and $f(b_{k-q-1}^i)$ must now be considered. When k is odd,

$$\begin{aligned} f(v_i) &= i(2n+1) - i - (2n+1) + 2 \\ &= 2n(i-1) + 1 \end{aligned}$$

for some $n \in \mathbb{Z}$, meaning $f(v_i)$ is always odd for $q = 2$. Because $q = 2$, by Remark 2.11 $f(v_i)$ and $f(b_{k-q-1}^i)$ are consecutive odd integers and are always relatively prime.

Therefore, $C_{k,2}^m$ with k odd has a cyclic snake prime labeling if and only if m is less than the smallest prime factor of $k-2$. □

2.4 Snakes with q = 3

Continuing to increase the q value, we next examine $q = 3$. By Theorem 2.9, we know snakes with cycles of length k, where k is a multiple of 3, are not prime. We now explore the snake graphs $C_{6n+4,3}^m$ and $C_{6n+5,3}^m$.

Theorem 2.14 *Let k be equivalent to* $4 \pmod 6$*. Then the cyclic snake labeling of the snake graph $C_{k,3}^m$ is prime if and only if $m \leq \left\lfloor \frac{p}{2} \right\rfloor$, where p is the smallest prime factor of $k-3$.*

Proof. Let p be the smallest prime factor of $k-3$, and note that p must be an odd prime. We first show that if $m > \left\lfloor \frac{p}{2} \right\rfloor$, then two adjacent vertices will share a common factor, and the cyclic snake labeling will not be prime. Consider $f\left(v_{\frac{p-1}{2}+1}\right)$ and $f\left(s_1^{\frac{p-1}{2}+1}\right)$.

By the cyclic prime labeling, for $1 \leq i \leq m$, $f(s_1^i) = ik - i - 1$ and $f(v_i) = ik - i - k + 2$. Let $i = \frac{p-1}{2} + 1$.

$$\begin{aligned} f\left(v_{\frac{p-1}{2}+1}\right) &= \frac{p-1}{2}k + k - \frac{p-1}{2} - 1 - k + 2 \\ &= \frac{p-1}{2}(k-1) + 1 \\ &= \frac{pk - k - p + 1 + 2}{2} \\ &= \frac{p(k-1) - (k-3)}{2} \end{aligned}$$

Similarly,

$$
\begin{aligned}
f\left(s_1^{\frac{p-1}{2}+1}\right) &= \frac{p-1}{2}k + k - \frac{p-1}{2} - 2 \\
&= \frac{p-1}{2}(k-1) + k - 2 \\
&= \frac{p-1}{2}(k-1) + (k-3) + 1 \\
&= \frac{(p-1)(k-1) + 2(k-3) + 2}{2} \\
&= \frac{pk - k - p + 1 + 2(k-3) + 2}{2} \\
&= \frac{pk - p - (k-3) + 2(k-3)}{2} \\
&= \frac{p(k-1) + (k-3)}{2}.
\end{aligned}
$$

Because p is an odd prime that divides $k-3$, p divides both $f\left(v_{\frac{p-1}{2}+1}\right)$ and $f\left(s_1^{\frac{p-1}{2}+1}\right)$. Therefore, these labels are not relatively prime, and the cyclic snake labeling is not prime when $m > \left\lfloor \frac{p}{2} \right\rfloor$.

Now, we show if $m \leq \left\lfloor \frac{p}{2} \right\rfloor$, then all adjacent vertices have relatively prime labels. By Remark 2.11, we only consider the non-consecutive labels, $f(v_i)$ and $f(s_1^i)$ as well as $f(b_{k-q-1}^i)$ and $f(v_{i+1})$ for $1 \leq i \leq m$.

Let us first consider $f(v_i)$ and $f(s_1^i)$. We assume d is the smallest positive integer such that $f(v_i) = dq_1$, $f(s_1^i) = dq_2$. By Remark 2.11, $f(v_i) - f(s_1^i) = k - 3$, implying that d also divides $k-3$.

Consider $f(s_1^i) = ik - i + 1 - 3 + 1 = ik - i - 1 = i(k-1)$. Because d divides $k-3$, it must be an odd prime. Therefore, it cannot also divide $k-1$ and must divide i. Hence $d \leq i < p$, and $d = 1$. Consequently, $f(s_1^i)$ and $f(v_i)$ are also relatively prime.

Similarly, we must show that the label of the preceding belly vertex is relatively prime to the adjacent spinal vertex. Observe that $f(b_{k-q-1}^i) - f(v_{i+1}) = 3$. Also, since $k \equiv 4 \pmod 6$,

$$
\begin{aligned}
f(v_{i+1}) &= (i+1)k - (i+1) - k + 2 \\
&= ik - i + 1 \\
&= i(k-1) + 1 \\
&= i(6n + 4 - 1) + 1 \\
&= 6in + 1 \equiv 1 \pmod 3.
\end{aligned}
$$

Since the label of $f(v_{i+1})$ is not divisible by 3, but 3 divides the absolute difference of $f(b_{k-q-1}^i)$ and $f(v_{i+1})$, it follows from Lemma 2.12 that the two labels are relatively prime.

Therefore, the snake graph $C^m_{k,3}$ has a cyclic snake prime labeling when k is equivalent to 4 (mod 6) if and only if $m \leq \left\lfloor \frac{p}{2} \right\rfloor$. $\square$

An example of Theorem 2.14, can be seen in Figure 8.

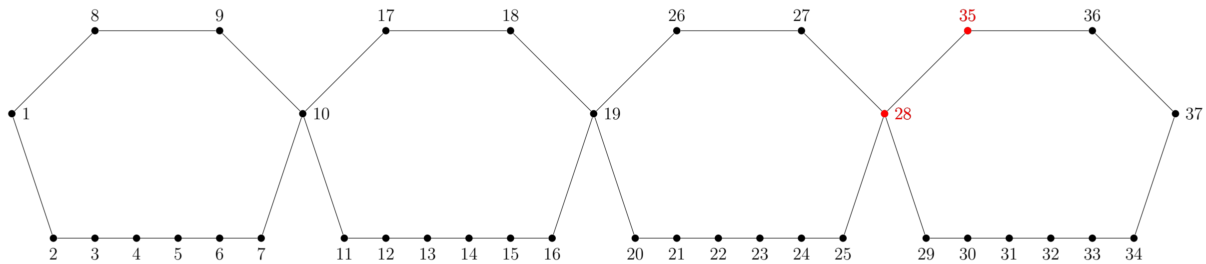


Figure 8: The snake graph $C^4_{10,3}$. As shown in Theorem 3.4, the cyclic snake labeling is a prime labeling only for the first three cycles of this snake as $k-3 = 10-3 = 7$ so $p = 7$ and $m \leq \left\lfloor \frac{7}{2} \right\rfloor = 3$. The fourth cycle shows adjacent labels that share a common factor.

Theorem 2.14 closely resembles what we observed for $q = 2$ snakes. The number of cycles contained in the snake graph with a cyclic prime labeling is dependent on the k value. However, when $q = 3$ and the k value is equivalent to 5 (mod 6), we find that the graph can only admit a cyclic snake prime labeling when $m = 1$, as shown in Theorem 2.15.

Theorem 2.15 *Let $k \equiv 5 \pmod 6$. Then the cyclic snake labeling of the snake graph $C^m_{k,3}$ is prime if and only if $m = 1$.*

Proof. Let p be the smallest prime factor of $k-3$, and note that p must be an odd prime. We first show that if $m \geq 2$, then two adjacent vertices will share a common factor, and the cyclic snake labeling will not be prime. Consider $f\left(v_{\frac{p-1}{2}+1}\right)$ and $f\left(s_1^{\frac{p-1}{2}+1}\right)$.

Let $C^m_{k,3}$ be a snake graph where k is odd and equivalent to 5 (mod 6). By the cyclic snake labeling, $f(v_3) = 3k-3-k-2 = 2k-1$ where $k = 6n+5$. By substitution,

$$\begin{aligned} f(v_3) &= 2(6n+5)-1 \\ &= 6(n+1)+3. \end{aligned}$$

The adjacent belly vertex, $f(b^2_{k-q-1}) = 2k-2-k+(k-q-1)+2 = 2k-q-1$. That is,

$$\begin{aligned} f(b^2_{k-q-1}) &= 2(6n+5)-3-1 \\ &= 6(2n+1). \end{aligned}$$

Since both $f(v_3)$ and $f(b^2_{k-q-1})$ are multiples of 3, the cyclic snake labeling is not prime when $m \geq 2$.

Now consider $C^1_{k,3}$, the case where $m = 1$. The second vertebra has label, $f(v_2) = k = 3n+2$. Its adjacent belly vertebra is assigned the label $f(b^1_{k-q-1}) = k-q = 3n-1$. These labels have a difference of 3, but neither is divisible by 3. Thus, by Lemma 2.12, $f(v_2)$ and $f(b^1_{k-q-1})$ are relatively prime.

Hence, $C^m_{k,3}$ has a cyclic snake prime labeling if and only if $m = 1$. □

Figure 9 illustrates an example of Theorem 2.15.

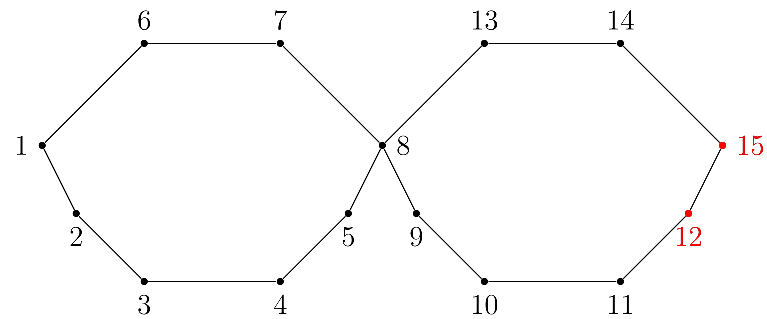


Figure 9: Snake $C^2_{8,3}$ is prime for one cycle. As indicated by the highlighted vertices, the cyclic snake labeling results in two adjacent labels sharing a common factor on the second cycle.

From Theorem 2.9, when $k \equiv 0 \pmod 6$ or $k \equiv 3 \pmod 6$, $C^m_{k,3}$ will not have a cyclic snake prime labeling. Above, we characterize the snakes $C^m_{k,3}$ having cycle lengths equivalent to $4 \pmod 6$ as well as those snakes having cycle lengths equivalent to $5 \pmod 6$. Thus, the cases where $k \equiv 1 \pmod 6$ or $k \equiv 2 \pmod 6$ are still open.

2.5 Snakes with q = 5

By Theorem 2.9, if a snake graph has a cycle length with many factors, then there are more restrictions on the q value in order for it to have a cyclic snake prime labeling. While we continue to increase our q value, we observe that snakes with even q values also have more restrictions, as we observed in Proposition 2.8 and Theorem 2.9. With this, we move from $q = 3$ to the next prime and analyze snakes with $q = 5$, $C^m_{k,5}$, where $k \equiv 1$ or $3 \pmod 5$.

Theorem 2.16 *If $k \equiv 1 \pmod 5$, the cyclic snake labeling of the snake graph $C^m_{k,5}$ is prime for all m.*

Proof. Let $k = 5n + 1$ for $n \in \mathbb{Z}^+$. By Remark 2.11, we need only consider the nonconsecutive labels, $f(v_i)$ and $f(s^i_1)$ as well as $f(v_{i+1})$ and $f(b^i_{k-q-1})$, for $1 \leq i \leq m$.

Let d be the smallest divisor of both $f(v_i)$ and $f(s^i_1)$. Then d must also divide their difference, $k - 5$, by Remark 2.11. Observe the vertex label for v_i is given by:

$$\begin{aligned}f(v_i) &= ik - i - k + 2\\ &= (ik - i - 2) - (k - 5).\end{aligned}$$

and the label for s_1^i is given by:

$$\begin{aligned}f(s_1^i) &= ik - i + 1 - 5 + 1\\ &= ik - i - 3\\ &= i(k-5) + (4i - 3).\end{aligned}$$

Since d divides $f(v_i)$, $f(s_1^i)$, and $k-5$, d must also divide both $ik - i - 2$ and $4i - 3$. Consequently, d must also divide $(ik - i - 2) - (4i - 3) = ik - 5i + 1 = i(k-5) + 1$. Because d divides $k-5$, d divides 1, and thus, $d = 1$. Therefore, the vertex labels $f(v_i)$ and $f(s_1^i)$ are relatively prime.

By Remark 2.11, the vertex labels $f(v_{i+1})$ and $f(b_{k-q-1}^i)$ are distance 5 apart. However, 5 does not divide $f(v_{i+1})$ since

$$\begin{aligned}f(v_{i+1}) &= (i+1)(k) - (i+1) - k + 2\\ &= (i+1)(k) - (i+1) - (k) + 2\\ &= (i+1)(k-1) - (k) + 2\\ &= (i+1)(5n) - (5n+1) + 2\\ &\equiv 3 \pmod 5.\end{aligned}$$

By Lemma 2.12, these two labels must be relatively prime.

Therefore we have proven that snake graphs $C_{k,5}^m$ have a cyclic prime labeling for all m when $k \equiv 1 \pmod 5$. □

As with the case for $C_{k,3}^m$ presented in Theorem 2.15, the snake graph $C_{k,5}^m$ can only contain one cycle when $k \equiv 3 \pmod 5$.

Theorem 2.17 *The cyclic snake labeling of the snake graph $C_{k,5}^m$ is prime for $k \equiv 3 \pmod 5$ if and only if $m = 1$.*

Proof. Assume $k \equiv 3 \pmod 5$. By the cyclic snake labeling and Remark 2.11, the label of v_3 is given by:

$$\begin{aligned}f(v_3) &= 2k - 1\\ &= 2(5n+3) - 1\\ &= 5(2n+1).\end{aligned}$$

The adjacent belly vertex, b^2_{k-q-1} is given by:

$$\begin{aligned} f(b^2_{k-q-1}) &= ik - i - 5 + 1 \\ &= 2(5n+3) - 2 - 5 + 1 \\ &= 10n. \end{aligned}$$

Since the label of the third vertebra, v_3, and its adjacent belly vertebra, b^2_{k-q-1}, both have a common factor of 5, the cyclic snake labeling of $C^m_{k,5}$ is not prime when $m \geq 2$.

Now, to show that the cyclic snake labeling is prime when $m = 1$, we only consider the first cycle. The only adjacent vertices that have non-consecutive, non-one labels are v_2 and b^1_{k-q-1}. By Remark 2.11, these labels are distance $q = 5$ apart. However, since $f(v_2) = k$ and 5 does not divide k, by Lemma 2.12 the two labels are both relatively prime. Since these labels are relatively prime, the cyclic snake labeling is prime for $C^1_{k,5}$, where $k \equiv 3 \pmod 5$. □

From Theorem 2.9, if $k \equiv 0 \pmod 5$, $C^m_{k,5}$ will not have a cyclic snake prime labeling. Above, we show that all snakes $C^m_{k,5}$ having cycle lengths equivalent to $1 \pmod 5$ have a cyclic snake prime labeling. We also show that snakes with cycle lengths equivalent to 3 (mod 5) only have a cyclic snake prime labeling when $m = 1$. Thus, the cases where k is equivalent to $2 \pmod 5$ and $4 \pmod 5$ are still open.

3 Modified Snake Labeling for q = 2

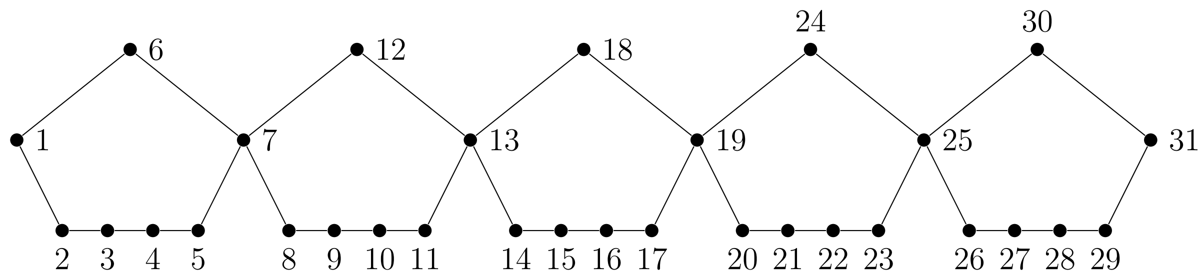
By Theorem 2.13, every snake graph $C^m_{k,2}$ has a prime labeling when m is less than the smallest prime factor of $k-2$, p. By following an adjusted labeling, we can modify the vertex labels of the p^{th} cycle which previously shared a factor, $f(v_p)$ and $f(s^p_1)$, so that these values are now no longer adjacent.

Definition 3.1 *The modified cyclic snake labeling for snakes $C^m_{k,2}$ is defined by the following bijective function. For any vertex x in $C^m_{k,2}$ such that $1 \leq j \leq k-q-1$, and $1 \leq \ell \leq q-1$, if $\gcd(m, k-2) = 1$, we label x by the original f labeling, so $F(x) = f(x)$.*

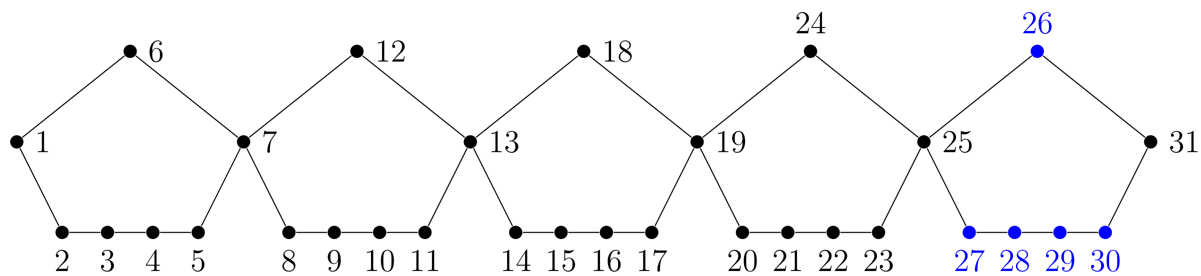
If $\gcd(m, k-2) \neq 1$, we have:

$$F(x) = \begin{cases} ik - i - k + 2 & \text{if } x = v_i \\ ik - i - k + 3 & \text{if } x = s^i_1 \\ ik - i - k + 3 + j & \text{if } x = b^i_j. \end{cases}$$

This modified labeling shifts labels from the cyclic labeling to produce adjacencies that are either consecutive integers or consecutive odd integers. An example of the original labeling as well as the modified labeling with the "shift" is shown in Figure 10.



(a) $C^5_{7,2}$ with the cyclic snake labeling $f(x)$.



(b) $C^5_{7,2}$ following the modified cyclic snake labeling $F(x)$. Differences between $f(x)$ and $F(x)$ are highlighted in blue.

Figure 10: Snake $C^5_{7,2}$. The original cycle snake labeling, $f(x)$, is given in (a) while the modified cyclic snake labeling, $F(x)$, is given in (b).

We present the following lemma which will assist us in proving the modified labeling, F, is a prime labeling of $C^m_{k,2}$.

Lemma 3.2 *Let k be an odd integer such that $k \geq 5$, and let p be the smallest prime factor of $k-2$. The integers $pk - p - k + 3$ and $(p+1)k - (p+1) - k + 2$ are relatively prime.*

Proof. Let k be an odd integer, and let p be the smallest prime factor of $k-2$. Let d be a positive integer that divides both p and $k-2$. Then there exist q_1 and q_2 such that $dq_1 = pk - p - k + 3$ and $dq_2 = (p+1)k - (p+1) - k + 2$. Observe:

$$\begin{aligned} dq_2 - dq_1 &= [(p+1)k - (p+1) - k + 2] - (pk - p - k + 3) \\ d(q_2 - q_1) &= (pk + k - p - 1 - k + 2) - (pk - p - k + 3) \\ &= (pk - p + 1) - (pk - p - k + 3) \\ &= pk - p + 1 - pk + p + k - 3 \\ &= k - 2. \end{aligned}$$

Thus, d divides $k-2$. Since p is the smallest prime factor of $k-2$, we can write $k - 2 = pc$ for some positive integer c. Since p is the smallest prime factor, c must be the product of primes greater than or equal to p.

Observe that:

$$\begin{aligned} pk - p - k + 3 &= (p-1)(k-1) + 2 \\ &= (p-1)(k-2) + (p-1) + 2 \\ &= (p-1)(k-2) + (p+1). \end{aligned}$$

Because d divides both the left-hand side of the equation and $k-2$, d must divide $p+1$. However, as $p+1$ is even, $p+1$'s prime factorization must be comprised of powers of 2 and other primes smaller than p. Since p is the smallest prime factor of $k-2$, d divides both $p+1$ and $k-2$ only if d is equal to 1. It follows that the only positive integer that divides both $pk-p-k+3$ and $(p+1)k-(p+1)-k+2$ is 1. Therefore, when k is odd and p is the smallest prime factor of $k-2$, $pk-p-k+3$ and $(p+1)k-(p+1)-k+2$ are relatively prime. □

We now show that the labeling F yields a prime labeling for $C_{k,2}^m$ when m is less than or equal to the smallest prime factor of $k-2$.

Theorem 3.3 *Let $k \geq 5$ be an odd integer, and let m be less than or equal to the smallest prime factor of $k-2$. Then the graph $C_{k,2}^m$, has a modified cyclic snake prime labeling.*

Proof. Let $C_{k,2}^m$ be a snake graph for some odd $k \geq 5$. Let p be the smallest prime factor of $k-2$, so $m \leq p$. If $m < p$, then by Theorem 2.13, the graph is prime.

Thus, it remains to consider the case where $m = p$. By Theorem 2.13, the first $m-1$ cycles must be prime, so we only consider the m^{th} cycle, or equivalently the p^{th} cycle.

First, we consider the fusing vertex v_p and the spine vertex s_1^p. The labels are given by $F(v_p) = pk-p-k+2$ and $F(s_1^p) = pk-p-k+3$. These are consecutive integers and are therefore relatively prime.

We next consider the vertebra v_p and the first belly vertex on the p^{th} cycle, b_1^p. The labels are given by $F(v_p) = pk-p-k+2$ and $F(b_1^p) = pk-p-k+3+1 = pk-p-k+4$. Since p and k are odd, $pk-p-k$ must be odd. These two labels are consecutive odd integers and, consequently, are relatively prime.

Observing the labels of the belly vertices, we observe that j increases by one between each belly vertex, so all of the belly vertices are consecutive and must be relatively prime.

We must also consider the relationship between the final belly vertex, b_{k-3}^p and the next fusing vertex s_{p+1}. The vertex labels are given by $F(b_{k-3}^p) = pk-p-k+3+k-3 = pk-p$, and $F(v_{p+1}) = (p+1)k-(p+1)-k+2 = pk-p+1$. These labels are consecutive integers and thus are relatively prime.

The final relationship to consider is the spine vertex s_1^p and the next vertebra, v_{p+1}. The two vertex labels are given by $F(s_1^p) = pk-p-k+3$ and $F(v_{p+1}) = (p+1)k-(p+1)-k+2$. By Lemma 3.2, these two labels are always relatively prime.

Therefore, every pair of adjacent vertex labels is relatively prime, and F is a prime labeling of $C_{k,2}^m$ when $m \leq p$, where p is the smallest prime factor of $k-2$. □

This modified labeling allows us to find a prime labeling of all snake graphs $C_{k,2}^m$, for $1 \leq m \leq p$ and where p is the smallest prime factor of $k-2$. In some cases, this labeling can be extended further than the p^{th} cycle, but in others, it cannot. For example, when $k = 23$, $k - 2 = 21$ and $p = 3$, on the $2p = 6^{th}$ cycle, the spine label is 112, and the next fusing vertex label is 133. Because $\gcd(112, 133) = 7$, this labeling of graph $C_{23,2}^{m}$ is not prime when $m \geq 6$.

4 Conclusion and Future Work

In this paper, we introduced new notation for a snake graph, creating a more general definition. After defining a cyclic snake labeling, we explored snake graphs with q values of 1, 2, 3 and 5 and determined whether our labeling produced a prime cyclic snake. Our work leads to several questions that could be explored in the future.

While we did explore several values of q for our snake graphs, we were unable to create broad generalizations beyond Theorem 2.9. Such broad generalizations became more complex as q increased, and subsequently, general results for a defined relationship between q and k values were elusive.

Question 4.1 Can the cyclic snake labeling be extended for any snake $C_{k,q}^{m}$ based on the relationship between k and q values?

In Section 3, we explored "shifting" labels of vertices and labeling the spine vertex before the belly vertices. This "swap" allowed us to extend the number of cycles that can have prime labelings in a modified cyclic way. However, we were not able to find the maximum number of cycles for which a variation of the cyclic snake prime labeling existed.

Question 4.2 What is the maximum m for which $C_{k,2}^{m}$, k is odd and greater than or equal to 3, has a cyclic snake prime labeling where either the spine vertices or belly vertices of each cycle are labeled first?

Our work is entirely concerned with snake graphs with constant k and q values. However, it may be possible to relate the label values to varying k or q values if the varying values were related to the i^{th} cycle, which would not vary. An example is illustrated in Figure 11.

Question 4.3 Is there a predictable prime labeling for a snake graph such that each cycle of the snake does not necessarily have the same k or q values?

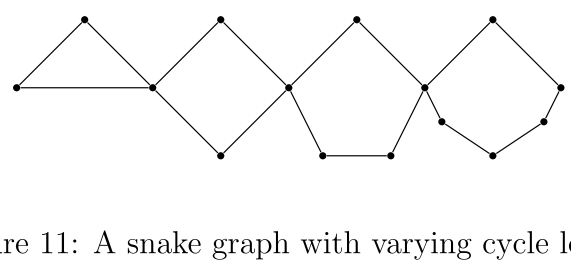


Figure 11: A snake graph with varying cycle lengths.

Coprime and minimal coprime labelings of graphs have been studied by Berliner et al. [1]. Coprime labelings are similar to prime labelings in that integers are assigned to the vertices so that any two adjacent vertices are relatively prime; however, the maximum vertex label is not limited by the number of vertices. Every graph can be labeled with all primes and is coprime, but there may be some way to label our snake graphs in a way that uses smaller numbers.

Question 4.4 For a snake graph that is not prime, what is the minimal coprime labeling?

There are numerous other labelings that could possibly be applied to snake graphs that we did not experiment with. Other types of labelings would be interesting to examine in the future.

Question 4.5 Can other labelings (e.g. neighborhood prime, graceful, magic, etc.) be applied to snake graphs?

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References

[1] A.H. Berliner, N. Dean, J. Hook, A. Marr, A. Mbirika, C.D. McBee, Coprime and prime labelings of graphs, *J. Integer Seq.*, **19** (2016), Article 16.5.8, 14 pp. Available online at the following URL: `https://cs.uwaterloo.ca/journals/JIS/vol19.html`

[2] T. Deretsky, S.-M. Lee, J. Mitchem, On vertex prime labelings of graphs, in *Graph theory, combinatorics, and applications, Vol. 1 (Kalamazoo, MI, 1988)*, 359–369, Wiley-Intersci. Publ., Wiley, New York, 1991.

[3] K. Eshighi, *Introduction to Graceful Graphs*, Sharif University of Technology, 2002. Available online at the following URL: `http://sharif.edu/~eshghi/GracefulGraphs-FinalEdition-89-12-15.pdf`

[4] J.A. Gallian, Graph Labeling, *Electron. J. Combin.*, Dynamic Survey 6. Available online at the following URL: `https://www.combinatorics.org/ojs/index.php/eljc/article/view/DS6`

[5] P. Haxell, O. Pikhurko, A. Taraz, Primality of trees, *J. Comb.*, **2** (2011), 481–500. Available online at the following URL: `https://www.intlpress.com/site/pub/pages/journals/items/joc/content/vols/0002/0004/a001/`

[6] A. Rosa, On certain valuations of the vertices of a graph, in *Theory of Graphs (Internat. Sympos., Rome, 1966)*, 349–355, Gordon and Breach, New York; Dunod, Paris, 1967.

[7] S.A. Schluchter, J.Z. Schroeder, K. Cokus, R. Ellingson, H. Harris, E. Rarity, T. Wilson, Prime labelings of generalized Petersen graphs, *Involve*, **10** (2017), 109–124.

[8] R. Tout, A.N. Dabboucy, K. Howalla, Prime labeling of graphs, *Natl. Acad. Sci. Lett.*, **5** (1982), 365–368.

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